Optics

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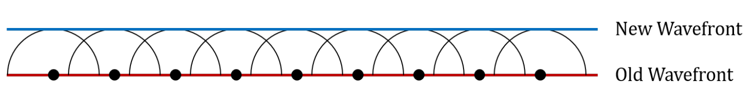
What follows is a quick recap of all the things needed for the first year Optics lecture series in the University of Oxford Physics course. All graphics are self-made.

Contents

1 Principles

Huygens' Principle

Light travels in wavefronts. Each new wavefront is created by forming a spherical wave at each point in the wavefront and then taking the envelope of these waves to be the new wavefront.



Remember that the angle a wavefront makes with a surface is the same as the angle the ray makes with the normal.

Fermat's Principle

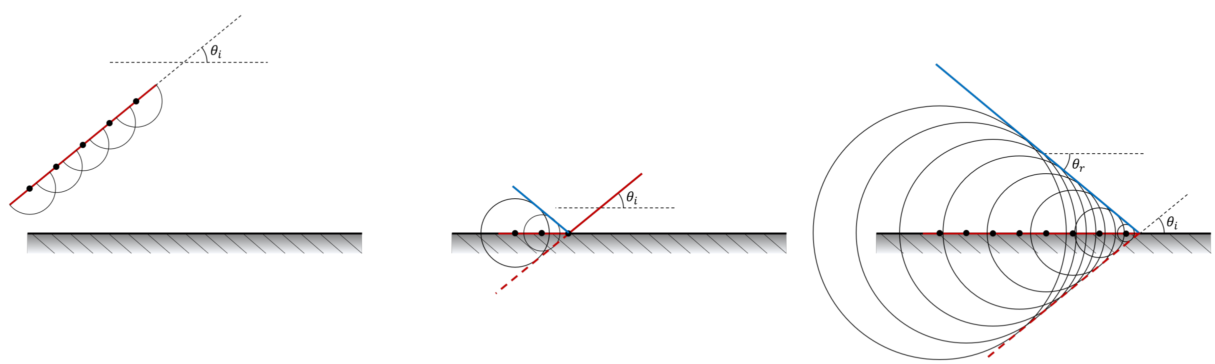
Light takes the path of stationary time. Often, this translates to the path of least time but is not always the case.

2 Phenomena

2.1 Reflection

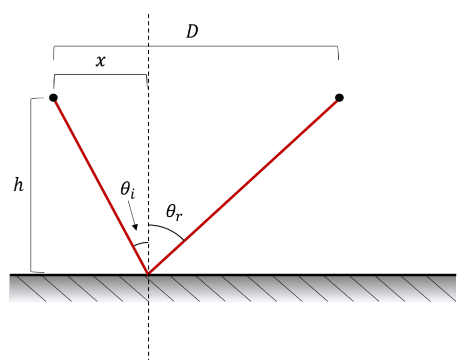
Take a light ray that hits a mirror, its angle to the normal is unchanged. This is **reflection**.

Proof with Huygens'



From the symmetry of the top and bottom in the last image, we know $\theta_i = \theta_r$.

Proof With Fermat's



h and D are constant. The total time of travel is

$$T = \frac{1}{v}\left(\sqrt{x^2+h^2} + \sqrt{(D-x)^2+h^2}\right) \tag{2.1}$$

To minimize this time by varying the point of contact to the mirror i.e. x, we make the following deduction.

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{1}{v}\left(\frac{x}{\sqrt{x^2+h^2}} - \frac{D-x}{\sqrt{(D-x)^2+h^2}}\right) \tag{2.2}$$

$$= \frac{1}{v}(\sin\theta_i - \sin\theta_r) \overset{!}{=} 0 \tag{2.3}$$

$$\implies \sin\theta_i = \sin\theta_r \tag{2.4}$$

$$\implies \theta_i = \theta_r \tag{2.5}$$

2.2 Refraction

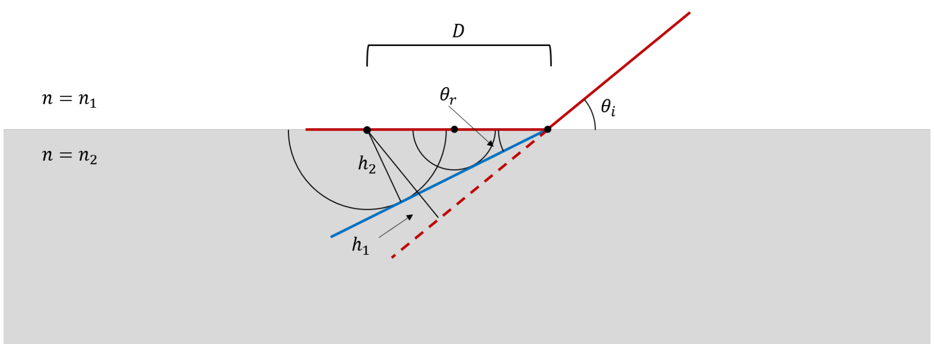
The **refractive index** n of a medium is related to the speed v in that medium and speed c in a vacuum by

$$n = \frac{c}{v} \tag{2.6}$$

Snell's Law of Refraction states that when light is refracting from a medium with refractive index n_1 to a medium with n_2, the angles to the normals of the incident ray θ_i and refracted ray θ_r by

$$n_1 \sin\theta_i = n_2 \sin\theta_r \tag{2.7}$$

Proof with Huygens'



From the diagram, we deduce

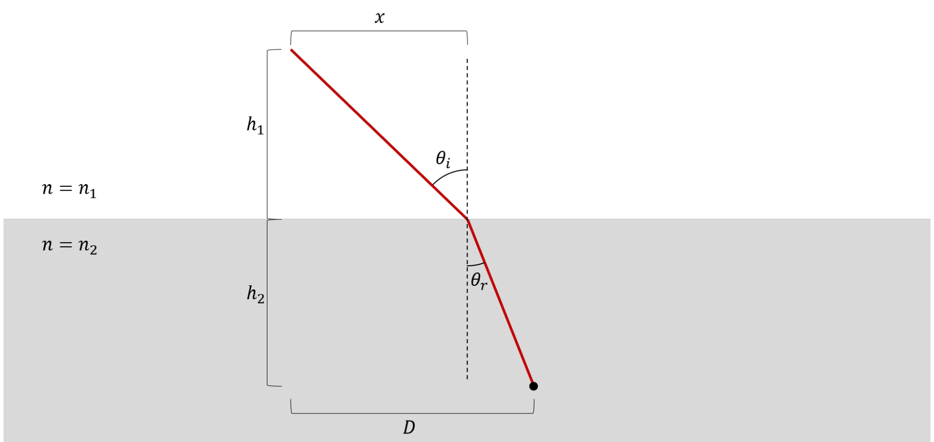
$$D\sin\theta_i = h_1 = v_1 t = \frac{ct}{n_1} \tag{2.8}$$

$$D\sin\theta_r = h_2 = v_2 t = \frac{ct}{n_2} \tag{2.9}$$

$$\implies D = \frac{ct}{n_1\sin\theta_i} = \frac{ct}{n_1\sin\theta_r} \tag{2.10}$$

$$\implies n_1\sin\theta_i = n_2\sin\theta_r \tag{2.11}$$

Proof With Fermat's



h_1, h_2, and D are constant. The total travel time is

$$T = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(D-x)^2 + h_2^2}}{v_2} = \frac{1}{c}\left(n_1\sqrt{x^2 + h_1^2} + n_2\sqrt{(D-x)^2 + h_2^2}\right) \tag{2.12}$$

Minimizing by varying x yields

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \frac{1}{c}\left(n_1\frac{x}{\sqrt{x^2 + h_1^2}} - n_2\frac{D-x}{\sqrt{(D-x)^2 + h_2^2}}\right) \tag{2.13}$$

$$= \frac{1}{c}(n_1 \sin\theta_i - n_2 \sin\theta_r) \tag{2.14}$$

$$\stackrel{!}{=} 0 \tag{2.15}$$

$$\tag{2.16}$$

$$\implies n_1 \sin\theta_i = n_2 \sin\theta_r \tag{2.17}$$

Total Internal Reflection

In reality, some light refracts inward and some light reflects at every surface. It is the balance of these two in any material. However when $\theta_r \geq \frac{\pi}{2}$, none refracts inward and there is **total internal reflection**. This condition yields a restriction on θ_i.

$$\theta_r = \frac{\pi}{2} \implies \theta_i = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{2.18}$$

This angle is called the **critical angle** denoted θ_c. For refraction,

$$\theta_i < \theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{2.19}$$

Note also that this only occurs when $n_2 < n_1$.

Optical Path Length

The **optical path length** OPL is defined as OPL $= nL$ where n is the refractive index of the medium and L is the actual length traveled. Useful for calculating phase differences of different paths.

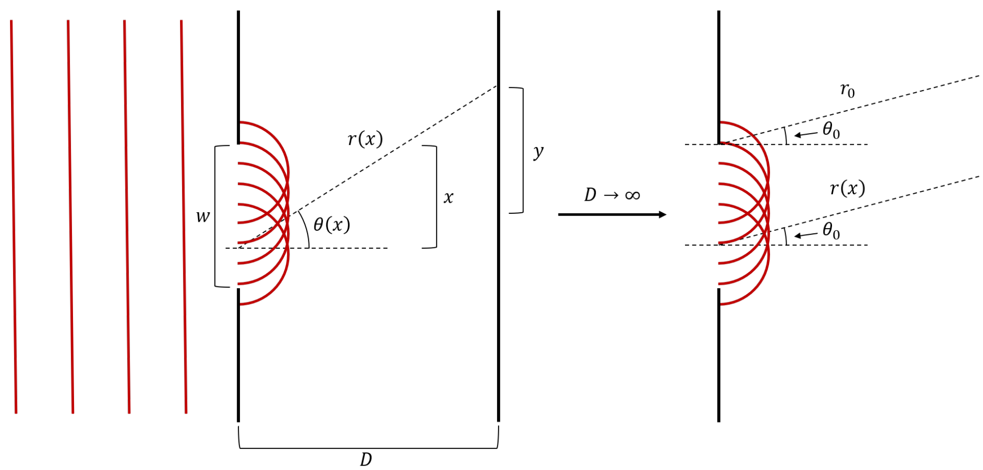
2.3 Diffraction (Fraunhofer)

For a slit of width w, the intensity as a function of the distance from the center y on an image plane a distance D away by a coherent λ wavelength light source normal to the plane is

$$I(y) = I_0 \frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2} \quad \text{where } \phi = \frac{2\pi w}{\lambda D} y \tag{2.20}$$

where I_0 is some constant and we assume $D \to \infty$. This is called **Fraunhofer diffraction**.

Proof with Huygens'



The differential Huygens wavelet at any point x along the slit is

$$\mathrm{d}\Psi = \alpha e^{i(kr-\omega t)} \mathrm{d}x \tag{2.21}$$

where $\alpha = \frac{A}{w}$.

When $D \to \infty$, from geometry, we know $r(x) = r_0 + x \sin\theta_0$. We can then integrate to find the total wave at that point.

$$\Psi = \int \mathrm{d}\Psi = \int_0^w \alpha e^{i(kr-\omega t)} \mathrm{d}x \tag{2.22}$$

$$= \int_0^w \alpha e^{i(kr_0+kx\sin\theta_0-\omega t)} \mathrm{d}x \tag{2.23}$$

$$= \alpha e^{i(kr_0-\omega t)} \int_0^w e^{ikx\sin\theta_0} \mathrm{d}x \tag{2.24}$$

$$= \frac{\alpha e^{i(kr_0-\omega t)}}{ik\sin\theta_0} \left(e^{ikw\sin\theta_0} - 1\right) \tag{2.25}$$

$$= \frac{2Ae^{i(kr_0-\omega t)}}{kw\sin\theta_0} \sin\left(\frac{kw\sin\theta_0}{2}\right) e^{i\frac{kw\sin\theta_0}{2}} \tag{2.26}$$

$$= A\frac{\sin\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)} e^{i\left(kr_0-\omega t)+\frac{kw\sin\theta_0}{2}\right)} \tag{2.27}$$

where $\phi = kw\sin\theta_0 = \frac{2\pi w}{\lambda D}y$. The intensity is then trivially calculated.

$$I = \Psi^*\Psi = A^2\frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2} = I_0\frac{\sin^2\left(\frac{\phi}{2}\right)}{\left(\frac{\phi}{2}\right)^2} \tag{2.28}$$

where $I_0 = A^2$.

Angular Resolution

The **angular resolution** θ_R is the ability of any image-forming device to distinguish small angles. Any two objects within an angle of θ_R cannot be distinguished.

Say we have some circular aperture of diameter D with wavelength λ light going through it, then the **Rayleigh criterion** says that the best angular resolution between two light sources is obtained when the first minimum of one diffraction pattern overlaps with the central maximum of the other. This results in the angular resolution

$$\theta_R \approx \frac{1.220\lambda}{D} \tag{2.29}$$

This is also the angular resolution of a simple lens.

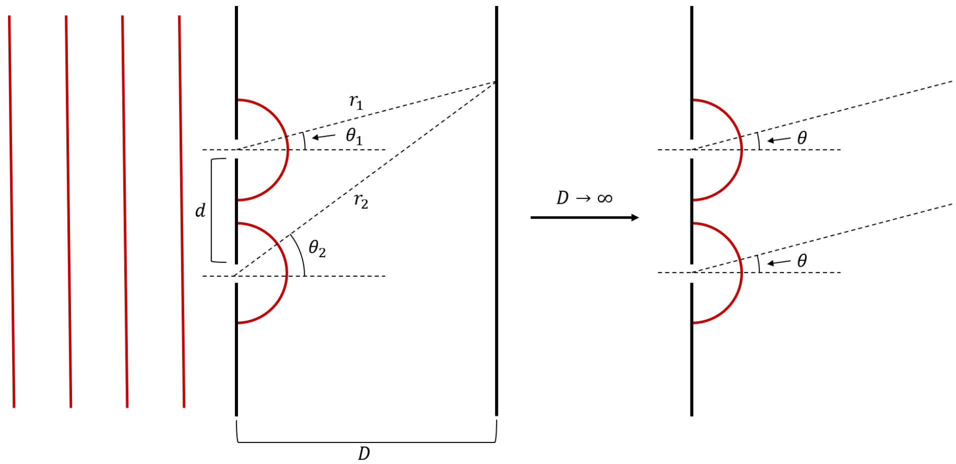
2.4 Interference (Two-Slit)

For two slits separated by a distance d, the intensity as a function of the distance from the center y on an image plane a distance D away by a coherent λ wavelength light source normal to the plane is

$$I(y) = 4I_0\cos^2\left(\frac{\phi}{2}\right) \quad \text{where } \phi = \frac{2\pi d}{\lambda D}y \tag{2.30}$$

where I_0 is some constant and we assume no diffraction. This is called **interference**.

Proof with Huygens'



The two waves at a point on the wall are

$$\Psi_1 = Ae^{i(kr_1 - \omega t)} \tag{2.31}$$
$$\Psi_2 = Ae^{i(kr_2 - \omega t)} \tag{2.32}$$

We then calculate intensity.

$$I = \Psi^*\Psi = (\Psi_1 + \Psi_2)^*(\Psi_1 + \Psi_2) \tag{2.33}$$
$$= |\Psi_1|^2 + |\Psi_2|^2 + 2\text{Re}[\Psi_1^*\Psi_2] \tag{2.34}$$
$$= 2A^2(1 + \text{Re}[e^{ik(r_2 - r_1)}]) \tag{2.35}$$

We know, from the geometry, that $r_2 - r_1 = d\sin\theta$ as $D \to \infty$.

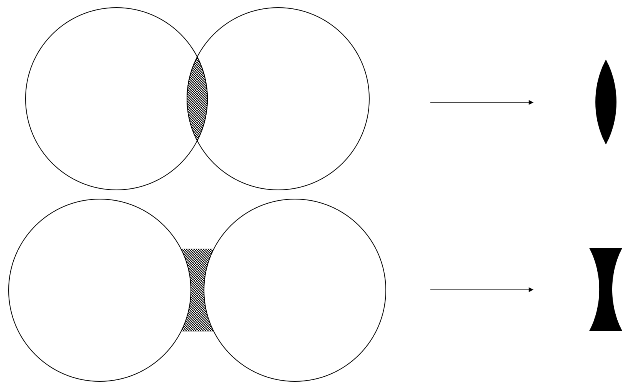
$$= 2A^2(1 + \text{Re}[e^{ikd\sin\theta}]) \tag{2.36}$$
$$= 2A^2(1 + \cos(kd\sin\theta)) \tag{2.37}$$
$$= 4I_0\cos^2\left(\frac{\pi d}{\lambda D}y\right) \tag{2.38}$$
$$= 4I_0\cos^2\left(\frac{\phi}{2}\right) \tag{2.39}$$

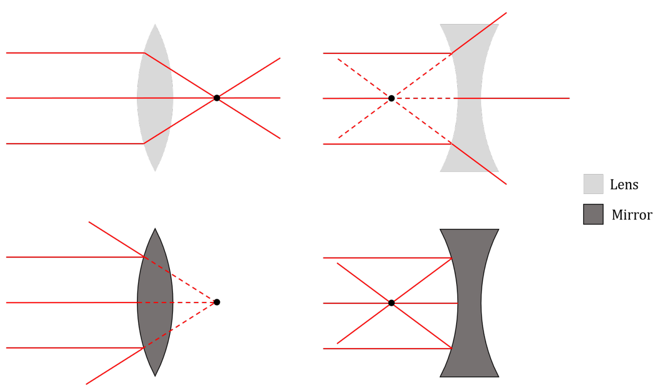
where $I_0 = A^2$.

3 Lenses and Mirrors

Note that all the following systems are in the **paraxial limit** i.e. assume small angles to ease ray tracing.

There are four main types of systems in geometric optics which are formed out of taking either the convex or concave form of a lens or mirror. They are created using the intersection of circles. All have a focal point f that rays from infinity form a real/virtual image at.





Note that the mirrors did not need the other side but I include them to show how closely related they are to the lenses. The concave lens is called **a thin lens**.

They are all related by the following equation for objects and images on the central axis.

$$\frac{1}{u} + \frac{1}{v} = \frac{1}{f} \tag{3.1}$$

where u gives the object position, v the image, and f the focus. For a plane mirror, $f \to \infty$. For the specific case of a thin lens, this is sometimes called the **thin lens equation** or **lens-makers formula**.

Now note the following facts.

Convention

- u is always positive and represents distance from center of lens/mirror
- $|v|$ represents the distance from center of lens/mirror
- v is $\begin{cases} \text{real} & \text{if } v > 0 \\ \text{virtual} & \text{if } v < 0 \end{cases}$
- Image on opposite side of object if lens and same if mirror

Focus

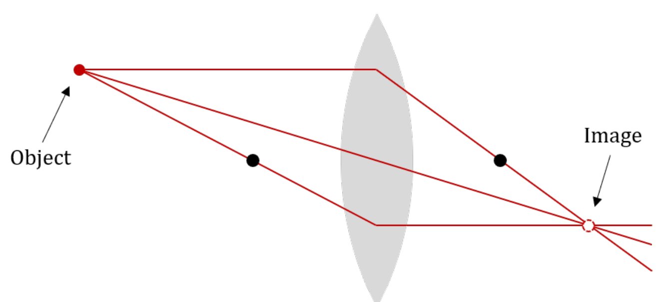
$$\frac{1}{f} = \begin{cases} (n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) & \text{if lens} \\ \frac{2}{r} & \text{if mirror} \end{cases} \tag{3.2}$$

where n is the refractive index of the lens, and r_1 and r_2 are the radii of the two circles used to create the lens. We assume the refractive index of outside the lens is approximately 1.

Geometric Ray Tracing

To find the image of off-axis points, we employ the following results.

- Rays from infinity (parallel to central axis) always go directly into or away from the focus like in the image of all four systems above.
- Rays through the focus always go out parallel to central axis
- Rays through the center are undeviated for a lens
- Rays at the center are reflected at the same angle to the normal for a mirror
- One can find the image of any point by drawing a ray coming out that is parallel to it and another that is through/at the center. By applying the principles above, one can find the resultant paths and trace them forward or backward to find their intersection, the image. A third ray directly through the focus could also be helpful for lenses. The following is an example.



Magnification

The magnification M is given by the following equation derived from similar triangles.

$$M = \frac{-v}{u} \tag{3.3}$$

Newtonian Form

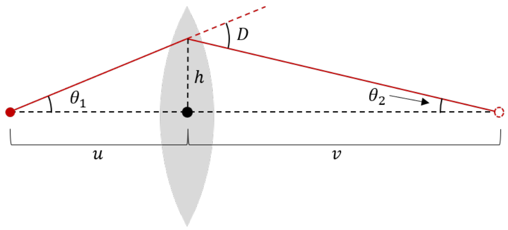
In Newtonian form, x_o and x_i are the distances of the object and image respectively from their foci i.e. $x_o = u - f$ and $x_i = v - f$. In this form,

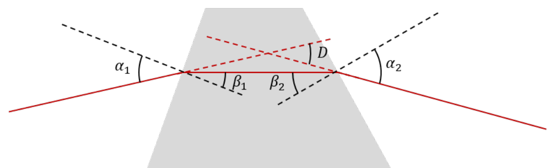
$$x_o x_i = f^2 \tag{3.4}$$

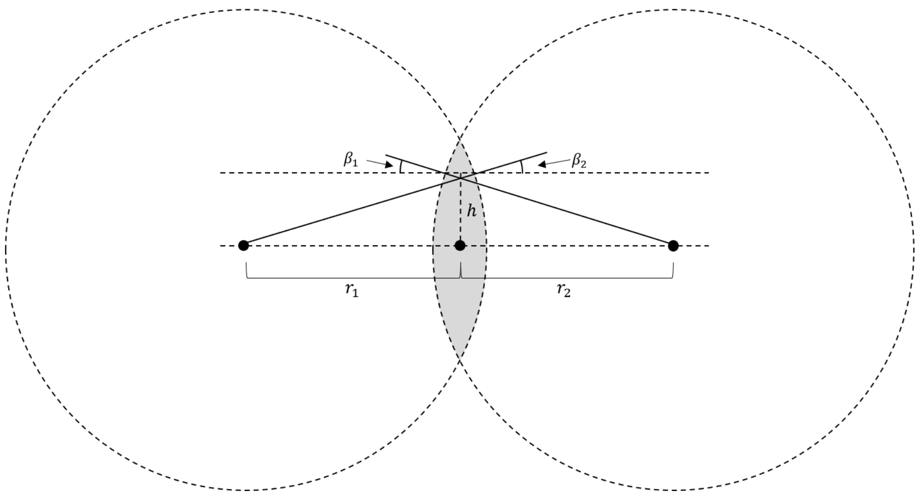
Thin Lens Formula Proof

I now prove $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$ for a thin concave lens.

Proof. Observe the following 3 images.







We deduce, from the first image, that

$$D = \theta_1 + \theta_2 \approx \frac{h}{u} + \frac{h}{v} \tag{3.5}$$

We select the path that results in a horizontal ray inside the lens. We then deduce, from the second image, that

$$\alpha_1 = n\beta_1 \tag{3.6}$$

$$\alpha_2 = n\beta_2 \tag{3.7}$$

$$\begin{aligned} D &= \alpha_1 - \beta_1 + \alpha_2 - \beta_2 && (3.8) \\ &= (n-1)(\beta_1 + \beta_2) && (3.9) \end{aligned}$$

We deduce, from the third image, that

$$\beta_1 \approx \frac{h}{r_2} \tag{3.10}$$

$$\beta_2 \approx \frac{h}{r_1} \tag{3.11}$$

Substituting into the last solution for D, we get

$$D = (n-1)\left(\frac{h}{r_2} + \frac{h}{r_1}\right) \tag{3.12}$$

We substitute this into our very first relation between the θ_i and D.

$$\frac{h}{u} + \frac{h}{v} = (n-1)\left(\frac{h}{r_2} + \frac{h}{r_1}\right) \tag{3.13}$$

$$\frac{1}{u} + \frac{1}{v} = (n-1)\left(\frac{1}{r_1} + \frac{1}{r_2}\right) \equiv \frac{1}{f} \tag{3.14}$$

□

Apex and Deviation Angle

The apex angle A (angle at top of thin lens) is related to the deviation angle D by $A = (n-1)D$.

Parabolic Mirrors

In reality, the spherical mirrors we describe only have the behavior with respect to the focus when they are parabolas but a circle is a good approximation of a parabola for small angles.