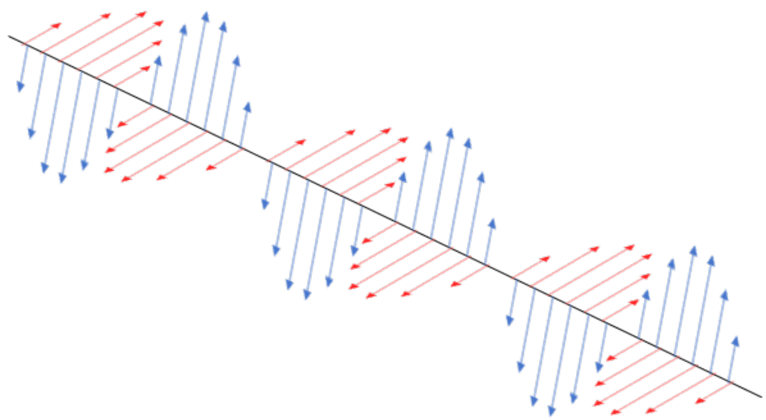
Waves



Aakash Lakshmanan

Balliol College, University of Oxford

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0 Introduction

The following is a set of lecture notes I made for first year Waves in the University of Oxford Physics course. To see the original notes by Professor Matt Jarvis, visit `https://www2.physics.ox.ac.uk/sites/default/files/2012-09-04/fullnotes2016_pdf_91657.pdf`.

These notes do however differ greatly in teaching style, organization of material, and level of mathematical rigor. They are written primarily for my friends who I wish the best in their endeavours in physics. If you like my writing and teaching style, feel free to look at my website (`https://xphysx.com/`) for more interesting physics and math. Enjoy!

1 Normal Modes

1.1 The Framework

Assume you have some mechanical system where the n positions of various objects are contained in a vector $\mathbf{x}$ and all the forces can be represented with a time-independent matrix Ω and matrix equation as follows.

$$\ddot{\mathbf{x}} = -\Omega \mathbf{x} \tag{1.1}$$

where $\ddot{\mathbf{x}} = \frac{\partial^2 \mathbf{x}}{\partial t^2}$.

Such an equation is simply a concise way of storing various smaller equations. One can imagine how it would naturally come up like, for example, in spring systems. Specific examples are given later which will also make clear the seemingly arbitrary negative sign.

How would we solve this? We first note require that K is diagonalizable (which is almost always the case for mechanical systems). Without this requirement, the methods following are mute. We then rely a lemma from linear algebra.

Lemma 1. The eigenvectors of any diagonalizable matrix M forms a basis.

Let $\mathbf{a}_i$ be the eigenvectors of Ω. Then from this lemma, we know they form a basis. This means $x(t)$ can be written as a linear combination of them.

$$\mathbf{x}(t) = \sum_i c_i(t)\mathbf{a}_i \tag{1.2}$$

We substitute into (1.1) denoting λ_i as the eigenvalues of Ω.

$$\sum_i \ddot{c}_i \mathbf{a}_i = -\Omega \left(\sum_i c_i \mathbf{a}_i \right) \tag{1.3}$$
$$= -\sum_i c_i (\Omega \mathbf{a}_i) \tag{1.4}$$
$$= -\sum_i \lambda_i c_i \mathbf{a}_i \tag{1.5}$$

Because the $\mathbf{a}_i$ are a basis, this implies

$$\ddot{c}_i = -\lambda_i c_i \tag{1.6}$$

We know the solution for this.

$$c_i(t) = A_i \cos \sqrt{\lambda_i} t + B_i \sin \sqrt{\lambda_i} t \tag{1.7}$$

Now, our final solution, plugging into (1.2), is

$$\mathbf{x}(t) = \sum_i \mathbf{a}_i \left(A_i \cos \sqrt{\lambda_i} t + B_i \sin \sqrt{\lambda_i} t \right) \tag{1.8}$$

There are $2n$ unknowns here (A_i and B_i) as expected from n second order differential equations.

Note: these methods can be applied to any linear matrix differential equation with some quick wit.

Example 1.1. Consider the situation in Figure 1.

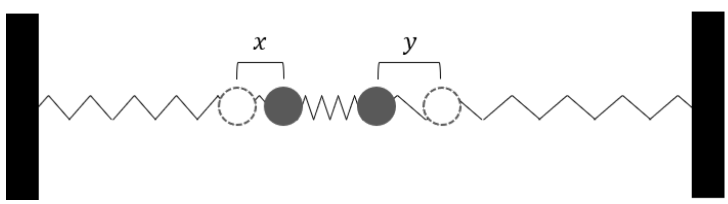

Figure 1: Two equal masses connected to two walls by springs of equal spring constant.

The equations for such a situation are

$$m\ddot{x} = k(y - x) - kx \tag{1.9}$$
$$m\ddot{y} = k(x - y) - ky \tag{1.10}$$

This means our matrix is

$$\Omega = \frac{k}{m} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \tag{1.11}$$

The eigenvalues of this matrix are $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$ with eigenvalues ω^2 and $3\omega^2$ respectively where $\omega = \sqrt{\frac{k}{m}}$. Applying our result, we get the following solution.

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} (A_1 \cos(\omega t) + B_1 \sin(\omega t)) + \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} (A_2 \cos(\sqrt{3}\omega t) - B_2 \sin(\sqrt{3}\omega t)) \tag{1.12}$$

1.2 Working in Normal Modes

We now pull from another lemma in linear algebra.

Lemma 2. For a diagonalizable matrix M whose eigenvectors $\mathbf{v}_i$ and eigenvalues λ_i are contained in $V := (\mathbf{v}_1, \ldots, \mathbf{v}_n)$ and $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)$,

$$M = V\Lambda V^{-1} \tag{1.13}$$

Let's apply this lemma. Let the eigenvectors and eigenvalues of Ω be contained in $A := (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ and $\Lambda := \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then, using this lemma, we can rearrange (1.1) in the following way

$$\begin{aligned} \ddot{\mathbf{x}} &= -\Omega \mathbf{x} && (1.14)\\ &= -A\Lambda A^{-1}\mathbf{x} && (1.15)\\ \implies A^{-1}\ddot{\mathbf{x}} &= -\Lambda A^{-1}\mathbf{x} && (1.16) \end{aligned}$$

Now, we define a useful idea.

Definition 1. The **normal modes** of Ω are defined as the components of the vector $\mathbf{u} = A^{-1}\mathbf{x}$ where $A = (\mathbf{a}_1, \ldots, \mathbf{a}_n)$ is the matrix of Ω eigenvectors.

So let us now write our equation above in terms of the normal modes $\mathbf{u}$.

$$\ddot{\mathbf{u}} = -\Lambda \mathbf{u} \tag{1.17}$$

Because Λ is diagonal, the rest of the solution is trivial.

$$\begin{aligned} \ddot{u}_i &= -\lambda_i u_i && (1.18)\\ \implies u_i &= A_i \cos\sqrt{\lambda_i}t + B_i \sin\sqrt{\lambda_i}t && (1.19) \end{aligned}$$

Each normal mode follows a path with just one frequency based on its corresponding eigenvalue. If we really wish to, we can then recover our original variables using our definition of u.

$$\mathbf{x} = A\mathbf{u} \tag{1.20}$$

The takeaway should be that every solution to the original matrix equation is just a linear combination of normal modes, each of which move at just one frequency.

Example 1.2. This definitely needs an example to do it justice. Let's revisit our previous spring example. We first form our A and Λ by calculating the eigenvectors and eigenvalues of Ω.

$$A = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \implies A^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{1.21}$$

$$\Lambda = \omega^2 \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \quad \text{where } \omega = \sqrt{\frac{k}{m}} \tag{1.22}$$

Our normal modes $\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}$ are then calculated from our original variables $\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}$.

$$\begin{pmatrix} u \\ v \end{pmatrix} = A^{-1}\mathbf{x} = \frac{1}{\sqrt{2}}\begin{pmatrix} y+x \\ y-x \end{pmatrix} \tag{1.23}$$

Then using our knowledge of normal modes, we know

$$\begin{pmatrix} u \\ v \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} A_1\cos(\omega t) + B_1\sin(\omega t) \\ A_2\cos(\sqrt{3}\omega t) + B_2\sin(\sqrt{3}\omega t) \end{pmatrix} \tag{1.24}$$

By adding or subtracting the two components of the equation, we see that this is equivalent to (1.12). Of course, removing the factor of $\sqrt{2}$ is fine as that can be absorbed into constants.

The appeal of this method however is the physical realization that the solution to these types of equations are simply sums of single frequency normal modes which, rather nicely in this example, was simply the sum and difference of the two positions. Intuitively, you can think of this as saying the distance between the two masses $(y-x)$ oscillates at a single frequency and the midpoint of the two masses ($\frac{1}{2}(L+y+x)$ where L is their distance at equilibrium) oscillates at another single frequency. Pretty amazing!

1.3 Quick Recap

- We wish to analyze the solutions to the matrix equation $\ddot{\mathbf{x}} = -\Omega\mathbf{x}$
- The solutions are
$$\mathbf{x}(t) = \sum_i \mathbf{a}_i\left(A_i\cos\sqrt{\lambda_i}t + B_i\sin\sqrt{\lambda_i}t\right)$$
where $\mathbf{a}_i$ and λ_i are the eigenvectors and eigenvalues respectively of Ω
- We can take linear combinations of our original variables to create the **normal modes** of the system. They are defined as the components of the vector
$$\mathbf{u} = A^{-1}\mathbf{x}$$
where $A := (\mathbf{a}_1, \ldots, \mathbf{a}_n)$
- The normal modes each oscillate in just one frequency
$$u_i = A_i\cos\sqrt{\lambda_i}t + B_i\sin\sqrt{\lambda_i}t$$

2 Standard Waves (Non-dispersive)

2.1 The Wave Equation

We now move on to an ubiquitous equation in physics. Take any system that has some classical wave-like variable in space and time $\Psi(\mathbf{x}, t)$. Then, the **wave equation** states that

$$\nabla^2\psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{2.1}$$

where c is the speed of waves in the medium. Note this is not true for *all* waves. Later, we discuss the many different fields it manifests itself in but it is important to know that this equation is the at the center of all classical wave theory. We discuss more complex waves that stray from this rule in Section 4.

For this course, we only concern ourselves with the one-dimensional wave equation.

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \Psi}{\partial t^2} \tag{2.2}$$

As we discuss the solutions and implications of this equation, it is helpful to have a strong visual in mind of what $\Psi = \Psi(x, t)$ actually represents. To visualize it, think of it as the height of string at some position and time or the string's shape at each moment in time. This is shown in Figure 2.

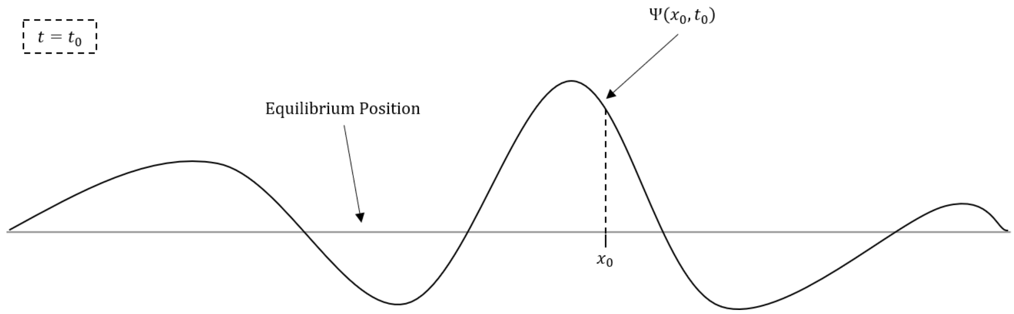

Figure 2: Demonstrating Ψ for a string. It shows the shape of the string at any point in time.

In Section 3, we look more deeply into the physics of this string system and how it obeys the wave equation. For now, we simply take it for granted and use it as purely a visual aid.

We solve the equation in three distinctly different but mathematically equivalent lights.

- **The General Wave Picture**: Solution as the sum of two general functions. Used primarily throughout the notes.
- **The Traveling Wave Picture**: Solution as the sum of traveling waves. A vital insight for understanding non-dispersive waves in Section 2.4.5
- **The Standing Wave Picture**: Solution as the sum of standing waves. Useful mathematical formulation for connecting to normal modes and solving problems of fixed/free ends which will come later.

2.2 The General Wave Picture

We employ a neat trick using the following variables (you will soon be very familiar with these two quantities).

$$u = x + ct \tag{2.3}$$

$$v = x - ct \tag{2.4}$$

Using chain rule and the fact that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 1$ and $\frac{\partial u}{\partial t} = -\frac{\partial v}{\partial t} = c$, we can compute the second derivatives of Ψ in terms of u and v. To avoid exhaustive equation grinding, I only show work for one but the other is very similar.

$$\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial \Psi}{\partial v}\frac{\partial v}{\partial t}\right) \tag{2.5}$$

$$= c\frac{\partial}{\partial t}\left(\frac{\partial \Psi}{\partial u} - \frac{\partial \Psi}{\partial v}\right) \tag{2.6}$$

$$= c\left(\frac{\partial}{\partial t}\frac{\partial \Psi}{\partial u} - \frac{\partial}{\partial t}\frac{\partial \Psi}{\partial v}\right) \tag{2.7}$$

$$= c\left(\frac{\partial^2 \Psi}{\partial u^2}\frac{\partial u}{\partial t} + \frac{\partial^2 \Psi}{\partial u \partial v}\frac{\partial v}{\partial t} - \frac{\partial^2 \Psi}{\partial v \partial u}\frac{\partial u}{\partial t} - \frac{\partial^2 \Psi}{\partial v^2}\frac{\partial v}{\partial t}\right) \tag{2.8}$$

$$= c^2\left(\frac{\partial^2 \Psi}{\partial u^2} - \frac{\partial^2 \Psi}{\partial u \partial v} + \frac{\partial^2 \Psi}{\partial v^2}\right) \tag{2.9}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial u \partial v} + \frac{\partial^2 \Psi}{\partial v^2} \tag{2.10}$$

Plugging (2.9) and (2.10) into the wave equation, we get the following.

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2}$$

$$\frac{\partial^2 \Psi}{\partial u^2} + \frac{\partial^2 \Psi}{\partial u \partial v} + \frac{\partial^2 \Psi}{\partial v^2} = \frac{1}{c^2}(c^2)\left(\frac{\partial^2 \Psi}{\partial u^2} - \frac{\partial^2 \Psi}{\partial u \partial v} + \frac{\partial^2 \Psi}{\partial v^2}\right) \tag{2.11}$$

Simplifying yields the following.

$$\frac{\partial^2 \Psi}{\partial u \partial v} = 0 \tag{2.12}$$

This is trivially solved.

$$\int \frac{\partial^2 \Psi}{\partial u \partial v} \mathrm{d}u = \int 0 \cdot \mathrm{d}u \tag{2.13}$$

$$\frac{\partial \Psi}{\partial v} = C(v) \tag{2.14}$$

$$\Psi = D(v) + E(u) \tag{2.15}$$

C,D, and E are arbitrary functions that are artifacts of integration. They are determined by initial conditions. We now have the solution to our wave equation (with renamed functions).

$$\Psi(x,t) = f(x - ct) + g(x + ct) \tag{2.16}$$

This is **d'Alembert's solution.**. It may not seem like a solution because f and g are still arbitrary but it reveals a very intrinsic connection between position and time. Let's look at just $f(x - ct)$. Consider the wave at some time t_0.

$$\Psi(x,t_0) = f(x-ct_0) \tag{2.17}$$

This means Ψ takes the shape of some arbitrary function but shifted to the right by ct_0 from our intuitive understanding of functions. As t increases, this offset will keep increasing and the function will appear to be moving rightward along the string. In fact this rightward motion will increase by ct or, in other words, at speed c. This is exactly what a wave is! By the same logic, we intuit that $g(x+ct)$ must be a left moving wave at speed c. The whole wave is simply the sum of these two waves moving in opposite directions. f and g merely determine the shape.

2.3 Initial Conditions

Arbitrary functions may still feel uncomfortable without a concrete solution to look at and gain intuition from. So let us investigate how we would form an actual solution given initial conditions. Let the initial state of the wave be $\Psi(x,0) = \Psi_0(x)$ and the initial velocity be $\Psi_t(x,0) = v_0(x)$. We put this into our solution for the wave.

$$\Psi(x,0) = f(x) + g(x) \stackrel{!}{=} \Psi_0(x) \tag{2.18}$$

$$\Psi_t(x,0) = -cf'(x) + cg'(x) \stackrel{!}{=} v_0(x) \tag{2.19}$$

$$\implies \int_b^x f'(x) - g'(x)\mathrm{d}x = -\frac{1}{c}\int_b^x v_0(x)\mathrm{d}x \tag{2.20}$$

$$\implies f(x) - g(x) = \frac{1}{c}\int_x^b v_0(x)\mathrm{d}x \tag{2.21}$$

We solve for f and g

$$f(x) = \frac{1}{2}\left(\Psi_0(x) + \frac{1}{c}\int_x^b v_0(x)\mathrm{d}x\right) \tag{2.22}$$

$$g(x) = \frac{1}{2}\left(\Psi_0(x) - \frac{1}{c}\int_x^b v_0(x)\mathrm{d}x\right) \tag{2.23}$$

You may be wondering what b is but don't worry, it cancels out. For now, it is simply an arbitrary constant. We have determined f and g so we plug back into our general Ψ solution.

$$\Psi(x,t) = f(x-ct) + g(x+ct) \tag{2.24}$$

$$= \frac{1}{2}\left(\Psi_0(x-ct) + \frac{1}{c}\int_{x-ct}^b v_0(x)\mathrm{d}x + \Psi_0(x+ct) - \frac{1}{c}\int_{x+ct}^b v_0(x)\mathrm{d}x\right) \tag{2.25}$$

$$= \frac{\Psi_0(x-ct) + \Psi_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(x)\mathrm{d}x \tag{2.26}$$

This is our final solution. For intuition, we consider the following example.

Example 2.1 (No Initial Velocity)**.** Let us consider a string that has no initial velocity. Mathematically,

$$v_0(x) = 0 \tag{2.27}$$

(2.26) then reads

$$\Psi(x,t) = \frac{\Psi_0(x-ct) + \Psi_0(x+ct)}{2} \tag{2.28}$$

It essentially says the initial wave splits into two waves, each with half the height and each moving in opposite directions. This is demonstrated with a simple square wave in Figure 3.

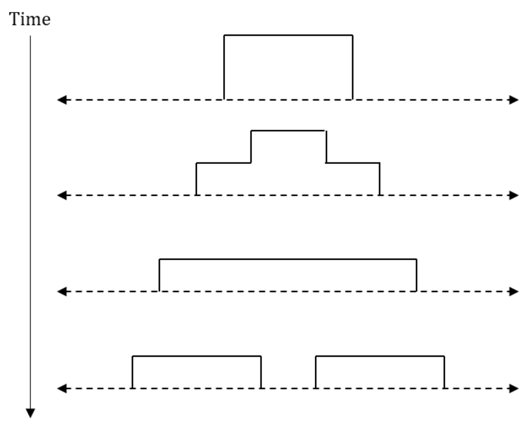

Figure 3: Demonstration of square wave evolving over time with no initial velocity.

2.4 The Traveling Wave Picture

We back up once again to the wave equation. This time, let's try guessing. We guess a cosine wave of the form $\Psi = A\cos(Bx + Ct + D)$. This is the most simplest wave in space and time you can get. Plugging this into the wave equation yields

$$\frac{\partial^2}{\partial x^2}(A\cos(Bx+Ct+D)) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(A\cos(Bx+Ct+D)) \tag{2.29}$$

$$-AB^2\cos(Bx+Ct+D) = -\frac{1}{c^2}AC^2\cos(Bx+Ct+D) \tag{2.30}$$

$$\implies B^2 = \frac{C^2}{c^2} \tag{2.31}$$

$$\implies C = \pm Bc \tag{2.32}$$

The only requirement is that the spatial coefficient B is related to the time coefficient C by (2.32).

Now that we know this yields a valid solution, let's assign a little more physical meaning to the constants before moving forward. We write B instead as k. k is called the **wave number**. Its physical significance comes from the fact that the cosine wave goes through one period in space when $Bx = kx$ increases by 2π or when x increases by $\frac{2\pi}{k}$. This means one period length or **wavelength** $\lambda = \frac{2\pi}{k}$. As wavelength is a more familiar concept, it may be more useful to think of the two concepts related by

$$k = \frac{2\pi}{\lambda} \tag{2.33}$$

We then write C as ω which is called the **angular frequency**. By using reasoning analogous to the wave number, we reason that ω is related to the **period** (in time) of the wave and, hence, the **frequency** f by

$$\omega = \frac{2\pi}{T} = 2\pi f \tag{2.34}$$

Often, when speaking of waves, although seemingly unnatural, only wave number and angular frequency are used.

D is just same phase offset usually written ϕ and A is just the amplitude written as is. Putting it all together, we have

$$\Psi = A\cos\left(kx \pm \omega t + \phi\right) \quad \text{where } \omega = ck \tag{2.35}$$

This is called a **traveling wave** for obvious reasons. It is a wave that looks like it is moving across the space over time. Use the same reasoning as the earlier sections to intuit why this is right/leftward moving.

Note that no restriction was created for the value of the wave number or angular frequency, only the relation between them. Because the differential equation is linear, this means the general solution must be a some over all possible wave numbers (as angular frequency is determined by wave number).

$$\Psi(x,t) = \int_{-\infty}^{\infty} A(k)\cos\left(kx + \omega t + \phi(k)\right)\mathrm{d}k \tag{2.36}$$

The takeaway from this picture is that all waves can be decomposed into traveling waves.

2.5 The Standing Wave Picture

We can find another equivalent class of solutions by considering the following method called **separation of variables**. Let the solution be written $\Psi(x,t) = X(x)T(t)$ or, in other words, let the function be separable into only a space-dependent and time-dependent component. We then deduce the following from the wave equation.

$$\frac{\partial^2}{\partial x^2}(XT) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2}(XT) \tag{2.37}$$

$$\implies X''T = \frac{1}{c^2}XT'' \tag{2.38}$$

$$\implies \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \tag{2.39}$$

One side of the equation is only space-dependent and the other only time. However, because they are always equal, this *must* mean they are a constant. Let's call it k^2. Then our two functions are

$$X(x) = A_1\cos\left(kx + \phi_1\right) \tag{2.40}$$

$$T(t) = A_2 \cos(kct + \phi_2) \tag{2.41}$$

We then have a solution substituting $A = A_1 A_2$

$$\Psi(x,t) = A\cos(kx + \phi_1)\cos(\omega t + \phi_2) \quad \text{where } \omega := ck \tag{2.42}$$

This is called a **standing wave**. The name exists as a reference to the fact that nothing seems to be moving left or right. At any given x_0, the term $\cos(kx_0 + \phi_1)$ is constant. Then the motion of the wave at that point is simply of the form $C\cos(\omega t + \phi_2)$. When seeing this visually, it seems like each point is simply oscillating at its own amplitude without any regard for the surrounding points. There is no apparent wave movement.

The general solution, like last section, is simply a sum over all possible k like earlier as it is a linear differential equation.

$$\int_0^\infty A(k)\cos(kx + \phi_1(k))\cos(\omega t + \phi_2(k))\mathrm{d}k \tag{2.43}$$

Note the intricacy where we ignored the possibility of the constant in (2.39) being negative or zero. Although these are actual solutions to the equation, we are only interested in the oscillatory finite ones. The others go off to infinity at the ends which is undesirable for a wave solution.

2.6 Connections

Coming soon! – We prove that all three pictures are mathematically equivalent. Essentially, general → traveling is Fourier Analysis and traveling → standing is two traveling waves opposite to each other.

2.7 Light and Sound

Let's look at a few examples of how the wave equation manifests in physics.

Light: Light is electromagnetic waves. Do these waves follow the same sort of form we have been analyzing? Well, let's write the Maxwell equations in a vacuum.

$$\nabla \cdot \mathbf{E} = 0 \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.44}$$

$$\nabla \cdot \mathbf{B} = 0 \quad \nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{2.45}$$

We now make the following deduction from these equations.

$$\begin{aligned}
\frac{\partial^2 \mathbf{E}}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{1}{\mu_0\epsilon_0}\nabla \times \mathbf{B}\right) && (2.46)\\
&= \frac{1}{\mu_0\epsilon_0}\nabla \times \left(\frac{\partial \mathbf{B}}{\partial t}\right) && (2.47)\\
&= \frac{-1}{\mu_0\epsilon_0}\nabla \times (\nabla \times \mathbf{E}) && (2.48)
\end{aligned}$$

We invoke the identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$.

$$= \frac{-1}{\mu_0\epsilon_0}(\nabla(\underbrace{\nabla \cdot \mathbf{E}}_{0}) - \nabla^2\mathbf{E}) \tag{2.49}$$

$$= \frac{1}{\mu_0\epsilon_0}\nabla^2\mathbf{E} \tag{2.50}$$

If we let $c = \sqrt{\frac{1}{\mu_0\epsilon_0}}$ and rearrange, we get

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{2.51}$$

The same can be done with the magnetic field.

$$\nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{2.52}$$

This is exactly our desired wave equation! The speed c that comes out is simply the speed of light. It then makes sense that light is just an electromagnetic wave.

Sound: Coming soon!

2.8 Quick Recap

- Waves, in classical physics, are usually described by the wave equation.

$$\frac{\partial^2\Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{2.53}$$

  where c is the speed of the wave in the medium

- The **general wave picture** show all solutions can be written with general functions.

$$\Psi(x,t) = f(x-ct) + g(x+ct) \tag{2.54}$$

  where f and g are arbitrary rightward and leftward moving waves respectively

- d'Alembert's solution, given initial position $\Psi_0(x)$ and velocity $v_0(x)$, is

$$\Psi(x,t) = \frac{\Psi_0(x-ct) + \Psi_0(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x-ct} v_0(x)\mathrm{d}x \tag{2.55}$$

- The **traveling wave picture** shows all solutions can be written as sums of **traveling waves**.

$$\int_{-\infty}^{\infty} A(k)\cos\left(kx + \omega t + \phi(k)\right)\mathrm{d}k \tag{2.56}$$

  where the **wave number** k and **angular frequency** ω are related to the wavelength λ and period T by

$$k = \frac{2\pi}{\lambda} \tag{2.57}$$

$$\omega = \frac{2\pi}{T} \tag{2.58}$$

- The **standing wave picture** shows all solutions can be written as sums of **standing waves**.

$$\int_0^\infty A(k)\cos(kx+\phi_1(k))\cos(\omega t+\phi_2(k))\mathrm{d}k \tag{2.59}$$

- All 3 pictures are mathematically equivalent.
- Light follows the wave equation in the form of changing electric **E** and magnetic **B** fields.

$$\nabla^2\mathbf{E} = \frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} \tag{2.60}$$

$$\nabla^2\mathbf{B} = \frac{1}{c^2}\frac{\partial^2\mathbf{B}}{\partial t^2} \tag{2.61}$$

where $c = \sqrt{\frac{1}{\mu_0\epsilon_0}}$ is the speed of light.

- Sound follows the wave equation in the form of changing pressure p in air.

$$\frac{\partial^2 p}{\partial x^2} = \frac{1}{c^2}\frac{\partial p}{\partial t} \tag{2.62}$$

where $c = \sqrt{\frac{B}{\rho}}$ is the speed of sound. B and ρ are the bulk modulus and mass density of the air.

3 A String

3.1 Connection to Wave Equation

Now let's consider the promised example. Imagine a string with mass density ρ and tension T running along the x axis. Let the string move in the y direction. Then, the string can be characterized by $y(x,t)$ which is the height of the string at any point along the string at a given time. Now, let's look at a single element of string as shown in Figure 4.

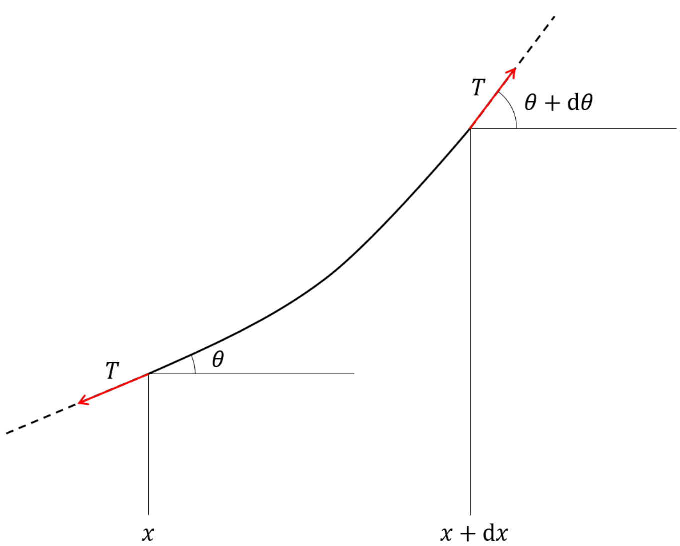

Figure 4: A differential element of a string.

The force on this segment is given by the two tensions in either direction but because the slope is slightly changing across the string, the direction of the tension does as well. As a result, we get a net force that looks like the following.

$$\mathbf{F} = \mathbf{F}_{\text{right}} + \mathbf{F}_{\text{left}} = T \begin{pmatrix} \cos(\theta + \mathrm{d}\theta) \\ \sin(\theta + \mathrm{d}\theta) \end{pmatrix} - T \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix} \tag{3.1}$$

$$= T \begin{pmatrix} \cos(\theta + \mathrm{d}\theta) - \cos(\theta) \\ \sin(\theta + \mathrm{d}\theta) - \sin(\theta) \end{pmatrix} \tag{3.2}$$

Now, we make the (uncomfortable) assumption that θ is small which means $\sin x \approx \tan x$ and $\cos x \approx 1$. Note that this is not actually terrible but requires the string to be at relatively low amplitudes.

$$= T \begin{pmatrix} 0 \\ \tan(\theta + \mathrm{d}\theta) - \tan(\theta) \end{pmatrix} = T(\tan(\theta + \mathrm{d}\theta) - \tan(\theta))\hat{\mathbf{y}} \tag{3.3}$$

The tangent is simply the slope $\frac{\partial y}{\partial x}$.

$$= T \left(\frac{\partial y}{\partial x}(x + \mathrm{d}x) - \frac{\partial y}{\partial x}(x) \right) \hat{\mathbf{y}} = T \mathrm{d}x \frac{\partial^2 y}{\partial x^2} \hat{\mathbf{y}} \tag{3.4}$$

We finish this by setting the force equal to $\mathbf{F} = (\mathrm{d}m)\mathbf{a}$ where $\mathrm{d}m$ is the mass of that string element and $\mathbf{a}$ is the acceleration. Well, no horizontal force means no horizontal acceleration or $\mathbf{a} = \frac{\partial^2 y}{\partial t^2}\hat{\mathbf{y}}$. This also means an interval of $\mathrm{d}x$ will always have the same amount of mass in it so $\mathrm{d}m = \rho \mathrm{d}x$. Putting this all together we get

$$\mathbf{F} = (\mathrm{d}m)\mathbf{a} = \rho \mathrm{d}x \frac{\partial^2 y}{\partial t^2} \hat{\mathbf{y}} \tag{3.5}$$

By setting (3.4) and (3.5) equal to each other, we get

$$T \mathrm{d}x \frac{\partial^2 y}{\partial x^2} \hat{\mathbf{y}} = \rho \mathrm{d}x \frac{\partial^2 y}{\partial t^2} \hat{\mathbf{y}} \tag{3.6}$$

$$\implies \frac{\partial^2 y}{\partial x^2} = \frac{\rho}{T} \frac{\partial^2 y}{\partial t^2} \tag{3.7}$$

$$\implies \frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \quad \text{where } c = \sqrt{\frac{T}{\rho}} \tag{3.8}$$

This is simply the wave equation where our speed is $c = \sqrt{\frac{T}{\rho}}$. All the previous results about the wave equation can now be applied to this string.

3.2 Energy

We now calculate the energy of these strings (the true mark of a physics course). As usual, we break this down into kinetic and potential energy. We start with kinetic.

Any string element has kinetic energy $\mathrm{d}K = \frac{1}{2}(\mathrm{d}m)v^2$. Using our previous argument about zero horizontal movement, we know this is equivalent to $\mathrm{d}K = \frac{1}{2}\rho \mathrm{d}x \left(\frac{\partial y}{\partial t}\right)^2$. Integrating across the string yields our kinetic energy.

$$K = \int_S \mathrm{d}K = \int_S \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \tag{3.9}$$

where S is the whole string.

Now for potential energy. Where is the potential energy coming from? It comes from the fact that the string becomes stretched by the tension T. Let a string element's length be $\mathrm{d}s$. The work the tension exerted on this element is simply force T times the distance it applied it for $(\mathrm{d}s - \mathrm{d}x)$. This means the energy of that segment $\mathrm{d}U$ is

$$\mathrm{d}U = T(\mathrm{d}s - \mathrm{d}x) \tag{3.10}$$

Take a moment to process why this would make sense. Now consider the actual definition of $\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2}$. We simplify it in the following way.

$$= T(\sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} - \mathrm{d}x) \tag{3.11}$$

$$= T\mathrm{d}x \left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} - 1\right) \tag{3.12}$$

Continuing off our assumption that our angles and amplitudes are small so our slopes are small, we take a low order Taylor approximation[1].

$$= T\mathrm{d}x \left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1\right) \tag{3.13}$$

$$= \frac{1}{2}T\mathrm{d}x \left(\frac{\partial y}{\partial x}\right)^2 \tag{3.14}$$

We, from here, trivially calculate the total potential energy.

$$U = \int_S \mathrm{d}U = \int_S \frac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \tag{3.15}$$

Our total energy is then

$$E = K + U = \int_S \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \tag{3.16}$$

This is admittedly not very pretty. But consider the following. Assume y is *only* right moving or left moving.

$$y(x,t) = f(x \pm ct) \tag{3.17}$$

Now consider it's derivatives.

[1]Note that the Taylor expansion used is $(1+x)^n \approx 1 + nx$ for small x. This is a *very* useful approximation and one that you should make natural to yourself.

$$\frac{\partial y}{\partial x} = f'(x \pm ct) \tag{3.18}$$

$$\frac{\partial y}{\partial t} = \pm c f'(x \pm ct) = \sqrt{\frac{T}{\rho}} f'(x \pm ct) \tag{3.19}$$

Then ...

$$\frac{\partial y}{\partial t} = \pm\sqrt{\frac{T}{\rho}}\frac{\partial y}{\partial x} \tag{3.20}$$

We substitute this into (3.9).

$$\begin{aligned}
K &= \int_S \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x && (3.21)\\
&= \int_S \frac{1}{2}\rho \left(\pm\sqrt{\frac{T}{\rho}}\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x && (3.22)\\
&= \int_S \frac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x && (3.23)\\
&= U && (3.24)
\end{aligned}$$

The kinetic energy of a string moving in only one direction is the same as its potential energy! Note that if it was moving in both directions, its time and spatial derivatives would not have been proportional making this deduction impossible. Now, its total energy is slightly simpler.

$$\begin{aligned}
E &= K + U && (3.25)\\
&= 2K = \int_S \rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x && (3.26)\\
&= 2U = \int_S T \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x && (3.27)
\end{aligned}$$

Example 3.1 (Average Energy of a Traveling Wave)**.** We now attempt to calculate the average energy per unit length of a traveling wave $y = A\cos(kx - \omega t)$. Let's first calculate the energy of one period. We do this by computing (3.26) from $x = 0$ to $x = \lambda$ which is the wavelength. From earlier we know $\lambda = \frac{2\pi}{k}$ so

$$\begin{aligned}
E_\lambda &= \int_0^{2\pi/k} \rho \left(\frac{\partial}{\partial t}(A\cos(kx - \omega t))\right)^2 \mathrm{d}x && (3.28)\\
&= \rho A^2\omega^2 \int_0^{2\pi/k} \sin^2(kx - \omega t)\mathrm{d}x && (3.29)\\
&= \rho A^2\omega^2 \frac{\pi}{k} && (3.30)
\end{aligned}$$

To get the average energy per unit length, we divide this by one wavelength.

$$\frac{\partial E}{\partial x} = \frac{E_\lambda}{\lambda} \tag{3.31}$$

$$= \frac{\rho A^2 \omega^2 \frac{\pi}{k}}{\frac{2\pi}{k}} \tag{3.32}$$

$$= \frac{1}{2}\rho A^2 \omega^2 \tag{3.33}$$

3.3 Boundary Problems: Reflection/Transmission

3.3.1 Conditions

Thus far, we have had only two variables to adjust in these strings: tension T and mass density ρ. We have kept them constant but now we will allow them to vary. More specifically, we will consider what happens when there is a sudden change in one. Consider a boundary at $x = 0$ with different mass density and tension on each side as shown in Figure 5.

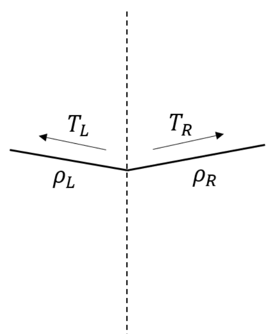

Figure 5: A point on a string with different mass densities and tensions.

Let the string to the left the boundary be described $y_L(x,t)$ and to the right by $y_R(x,t)$. Consider what must be true at that point.

The string must be continuous.

$$y_L(0,t) \stackrel{!}{=} y_R(0,t) \tag{3.34}$$

Now consider the vertical force at exactly that point. We know, from Section 3.1, that for low amplitudes, $\mathbf{F_y} = \pm T\left(\frac{\partial y}{\partial x}\right)$ where $\pm$ depends on the side. But, we also know there is virtually no mass at exactly the boundary so there can be no acceleration and, as a result, no force. The force equation then becomes

$$\mathbf{F}_y = T_R\left(\frac{\partial y_R}{\partial x}\right) - T_L\left(\frac{\partial y_L}{\partial x}\right) \stackrel{!}{=} 0 \tag{3.35}$$

$$\implies T_R\left(\frac{\partial y_R}{\partial x}\right) = T_L\left(\frac{\partial y_L}{\partial x}\right) \tag{3.36}$$

Note that we did not say explicitly that the x component of the force is zero. In fact, its not because the tensions differ which could cause some horizontal force. Not only is this impossible because of the zero mass at the center but this completely disintegrates our firm grasp of the wave equation which relies on no horizontal force. This problem however can simply be solved by adding a bar and a zero-mass ring at the center restricting horizontal motion while keeping the vertical uninhibited. This is shown in Figure 6.

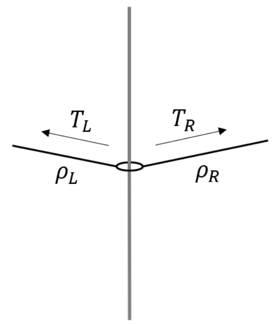

Figure 6: A point on a string with different mass densities and tensions but a massless ring attached to a vertical pole to prevent horizontal movement.

Now that we understand the constraints of the system, we can look to solve it.

3.3.2 Solving (with Impedance)

We attempt to understand the situation by seeing what happens to a wave that hits the boundary. Consider a wave to the left of the boundary traveling to the right i.e. $f(x - c_L t)$. Note that we say c_L because speed depends on the mass density and tension which changes based on side. Now, we ask the question: what happens at the boundary? The safest assumption is that there is a wave emitted in both directions or, in other words, a reflected and transmitted wave. This means the left side is a combination of a right-moving incident f_I and left-moving reflected f_R and the right side is just a right-moving transmitted f_T.

$$y_L = f_I(x - c_L t) + f_R(x + c_L t) \tag{3.37}$$

$$y_R = f_T(x - c_R t) \tag{3.38}$$

Remember that the only known function is f_I.

In our solutions, it turns out to be much much more convenient to write the solutions in the following forms. They are mathematically equivalent to the ones above.

$$y_L = f_I\left(t - \frac{x}{c_L}\right) + f_R\left(t + \frac{x}{c_L}\right) \tag{3.39}$$

$$y_R = f_T\left(t - \frac{x}{c_R}\right) \tag{3.40}$$

We first apply the continuity condition.

$$y_L(0,t) = y_R(0,t) \tag{3.41}$$

$$\implies f_I(t) + f_R(t) = f_T(t) \tag{3.42}$$

Now we apply the vertical force condition at $x = 0$.

$$T_L\left(\frac{\partial y_L}{\partial x}\right) = T_R\left(\frac{\partial y_R}{\partial x}\right) \tag{3.43}$$

$$\implies -\frac{T_L}{c_L} f_I'\left(t - \frac{x}{c_L}\right) + \frac{T_L}{c_L} f_R'\left(t + \frac{x}{c_L}\right) = -\frac{T_R}{c_R} f_T'\left(t - \frac{x}{c_R}\right) \quad \text{at } x = 0 \tag{3.44}$$

$$\implies -\frac{T_L}{c_L}(f_I'(t) - f_R'(t)) = -\frac{T_R}{c_R} f_T'(t) \tag{3.45}$$

$$\implies \frac{T_L}{c_L}(f_I(t) - f_R(t)) = \frac{T_R}{c_R} f_T(t) \tag{3.46}$$

To make the equations a little nicer, we define a new term which has very important physical implications.

Definition 2. For a string of tension T and wave speed c, the **impedance** is

$$Z = \frac{T}{c}$$

If string density ρ is given, this is also equivalent to $Z = \sqrt{T\rho}$.

Our equation is then fairly simple.

$$\implies f_I(t) - f_R(t) = \frac{Z_R}{Z_L} f_T(t) \tag{3.47}$$

Solving (3.42) and (3.46) for each function results in

$$f_R = \left(\frac{Z_L - Z_R}{Z_L + Z_R}\right) f_I \qquad f_T = \left(\frac{2Z_L}{Z_L + Z_R}\right) f_I \tag{3.48}$$

If we let $r = \frac{Z_L - Z_R}{Z_L + Z_R}$ and $t = \frac{2Z_L}{Z_L + Z_R}$, then

$$f_R = r f_I \qquad f_T = t f_I \tag{3.49}$$

where these numbers are called the **reflection coefficient** and **transmission coefficient** respectively.

From this, we now know that the reflected and transmitted waves have a the same general shape as the incident wave but with different heights based on the coefficients. In the case of the transmitted wave, the wave is also shorter/longer because it moves at a different speed from the incident.

Consider 3 cases.

$Z_L = Z_R$: This means $r = 0$ and $t = 1$ i.e. the wave is just transmitted. If the two sides are the same, this is clear but this statement implies that if you scale tension and mass density properly so as to keep the impedance constant, then the wave will still be fully transmitted.

$Z_L \gg Z_R$: This means $r = 1$ and $t = 2$. This happens when the string on the other side has essentially no tension or mass relative to the incident side. The wave is reflected back the exact same and transmits at twice the amplitude. Using the terminology of Section 3.5 (a later section), this is equivalent to a free end because the virtually zero impedance from the other side allows for free movement.

$Z_L \ll Z_R$: This mean $r = -1$ and $t = 0$. This situation represents when the transmitted side has seemingly infinite tension/mass causing the wave to just reflect back upside down with no transmission. Once again looking ahead to Section 3.5, this is physically equivalent to a static fixed end.

Note: The reflection and transmission coefficients *exactly* correspond to an elastic collision where a particle of initial mass Z_L and speed f_I collides with a particle at rest with mass Z_R. The final speed of the first and second particle are then f_R and f_T respectively. Pretty neat! Might be a nice way to help you remember it.

Example 3.2 (Constant Mass Density)**.** Let there be constant mass density ρ. Then our coefficients become

$$r = \frac{Z_L - Z_R}{Z_L + Z_R} = \frac{\sqrt{T_L\rho} - \sqrt{T_R\rho}}{\sqrt{T_L\rho} + \sqrt{T_R\rho}} = \frac{\sqrt{T_L} - \sqrt{T_R}}{\sqrt{T_L} + \sqrt{T_R}} \tag{3.50}$$

$$t = \frac{2Z_L}{Z_L + Z_R} = \frac{2\sqrt{T_L\rho}}{\sqrt{T_L\rho} + \sqrt{T_R\rho}} = \frac{2\sqrt{T_L}}{\sqrt{T_L} + \sqrt{T_R}} \tag{3.51}$$

Example 3.3 (Constant Tension)**.** Let there be constant tension T. Note that we can remove the ring for this case. Then our coefficients become

$$r = \frac{Z_L - Z_R}{Z_L + Z_R} = \frac{\sqrt{T\rho_L} - \sqrt{T\rho_L}}{\sqrt{T\rho_L} + \sqrt{T\rho_R}} = \frac{\sqrt{\rho_L} - \sqrt{\rho_R}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{3.52}$$

$$r = \frac{2Z_L}{Z_L + Z_R} = \frac{2\sqrt{T\rho_L}}{\sqrt{T\rho_L} + \sqrt{T\rho_R}} = \frac{2\sqrt{\rho_L}}{\sqrt{\rho_L} + \sqrt{\rho_R}} \tag{3.53}$$

3.3.3 Boundary Mass

Coming Soon!

3.4 Power Flow

Coming Soon!

3.5 Fixed/Free End Problems

We now look at a certain interesting class of problems. Consider the following constraints on a string: a **free end** where we cut the string off at a certain point and attach it to a ring on a vertical pole like before and a **fixed end** where the string is held at $y = 0$. Both these types of constrained ends are demonstrated in Figure 7.

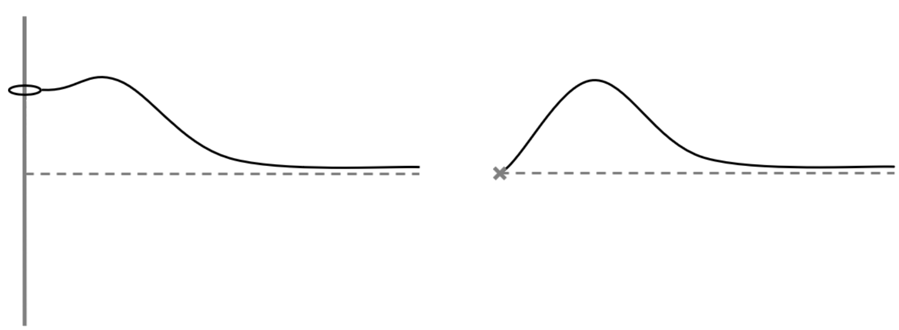

Figure 7: A demonstration of a free (left) and fixed (right) end.

We can imagine various combinations of these restrictions on a string (fixed and free, two free, one fixed, etc). To solve these problems, it is extremely useful to use the standing waves picture.

Let us recall what the standing wave picture tells us. Every wave can be written as the infinite sum over all possible spatial frequencies k of standing waves, or mathematically phrased

$$y(x,t) = \int_0^\infty A(k)\cos(kx + \phi_1(k))\cos(\omega t + \phi_2(k))\mathrm{d}k \tag{3.54}$$

Let's assume, without loss of generality, that the string starts at rest i.e. $\frac{\partial y}{\partial t} = 0$, then

$$\begin{aligned}
\frac{\partial y}{\partial t}(x, t=0) &= \int_0^\infty -\omega A(k)\cos(kx + \phi_1)\sin(\omega t + \phi_2)\mathrm{d}k \quad \text{at } t = 0 && (3.55)\\
&= \int_0^\infty -\omega A(k)\cos(kx + \phi_1)\sin(\phi_2)\mathrm{d}k && (3.56)\\
&\overset{!}{=} 0 && (3.57)
\end{aligned}$$

This requirement holds for all k and all x so we then get the condition

$$\begin{aligned}
\sin(\phi_2) &= 0 && (3.58)\\
\implies \phi_2(k) &= 0, \pi && (3.59)
\end{aligned}$$

We choose 0 without loss of generality as the negative sign that arises from choosing the π can be absorbed into $A(k)$. Our nicer formulation then looks like

$$y(x,t) = \int_0^\infty A(k)\cos(kx + \phi_1(k))\cos(\omega t)\mathrm{d}k \tag{3.60}$$

Now, let's investigate what happens when we constrain an end.

Fixed End at $x = x_0$:

Fixing the string to $y = 0$ at $x = x_0$ at all times simply means $A(k)\cos(kx + \phi_1) = 0$ for all k at $x = x_0$. Instead of solving for ϕ_1, we use some intuition about waves. Cosine and sine waves only differ by phase so let's just change to sine. $\sin(kx)$ has a 0 at the origin and $\sin(k(x - x_0))$ has a zero at $x = x_0$. Because this satisfies the constraints, we use it as the position component.

$$y(x,t) = \int_0^\infty A(k)\sin(k(x - x_0))\cos(\omega t)\mathrm{d}k \tag{3.61}$$

All standing waves with just this fixed end can be written in that form.

Free End at $x = x_0$

We know, from earlier, that tension and slope together result in vertical force but any force on a massless ring would result in infinite acceleration. There is tension so slope must be 0 at the ring. By noting the cosine has a slope of 0 at the origin and using similar intuition to the last section, we can then deduce that

$$y(x,t) = \int_0^\infty A(k)\cos(k(x - x_0))\cos(\omega t)\mathrm{d}k \tag{3.62}$$

We already start to see the convenience of standing wave picture in these constrained end scenarios. They are the sum of various standing waves with the same spatial phase offset. It becomes more interesting however when we look at two constrained ends.

Two Constrained Ends: For simplicity, let's assume our constrained ends are at $x = 0$ and $x = L$ i.e. the string has equilibrium length L. Let both be fixed. Then the result is the following

$$x = 0 \text{ Fixed} \iff y(0,t) = 0 \quad \forall k, t \tag{3.63}$$
$$\implies \cos(k \cdot 0 + \phi_1(k)) = 0 \tag{3.64}$$
$$\implies \phi_1(k) = \pi \left(\frac{1}{2} + r\right) \quad \forall r \in \mathbb{N} \tag{3.65}$$

$$x = L \text{ Fixed} \iff y(L,t) = 0 \quad \forall k, t \tag{3.66}$$
$$\implies \cos(kL + \phi_1(k)) = 0 \tag{3.67}$$
$$\implies kL + \phi_1(k) = \pi \left(\frac{1}{2} + s\right) \quad \forall s \in \mathbb{N} \tag{3.68}$$

We now have the two conditions (3.65) and (3.68). Subtracting one from the other, we get

$$kL = \pi(s - r) \quad \forall r, s \in \mathbb{N} \tag{3.69}$$
$$\implies k = \frac{\pi r}{L} \quad \forall r \in \mathbb{N} \tag{3.70}$$

This shows k is quantized. k can no longer take on all values between 0 and ∞. It now takes on integral values scaled by π/L. Physically, this would make sense as only certain frequencies would be able to fit the strict criteria of being 0 in exactly two places. These allowed spatial frequencies are called the **resonant frequencies**.

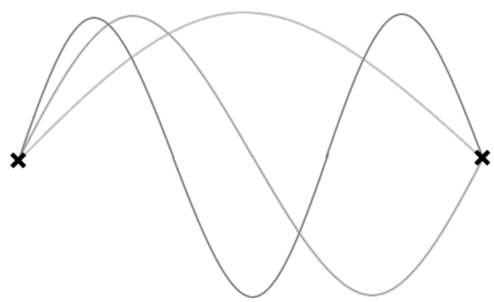

Figure 8: Two fixed ends and the first 3 normal modes.

Applying this condition to our original fixed end solution in (3.61),

$$y(x,t) = \int_0^\infty A(k) \sin(kx) \cos(\omega t) \mathrm{d}k \tag{3.71}$$
$$= \sum_{r=0}^{\infty} A_r \sin\left(\frac{\pi r}{L} x\right) \cos\left(\frac{\pi r c}{L} t\right) \tag{3.72}$$

Note that the integral is now just a infinite series. The different terms in this series are called the **normal modes** and reasonably so. Just like earlier, the normal modes are a list of solutions, each of which move at a single frequency and can be linearly combined to create all solutions.

You may wonder what the result would be for a free end. Well, consider the constraint of a free end at $x = 0$.

$$x = 0 \text{ Free} \iff \frac{\partial y}{\partial x}(0, t) = 0 \quad \forall k, t \tag{3.73}$$

$$\implies \sin(k \cdot 0 + \phi_1(k)) = 0 \tag{3.74}$$

$$\implies \phi_1(k) = \pi r \quad \forall r \in \mathbb{N} \tag{3.75}$$

Repeating the process of subtracting with the condition at $x = L$ and plugging back into our integral like with the two fixed ends, we get an answer very similar to before.

$$k = \frac{\pi \left(r + \frac{1}{2}\right)}{L} \quad \forall r \in \mathbb{N} \tag{3.76}$$

$$y(x, t) = \sum_{r=0}^{\infty} A_r \cos\left(\frac{\pi \left(r + \frac{1}{2}\right)}{L} x\right) \cos\left(\frac{\pi \left(r + \frac{1}{2}\right) c}{L} t\right) \tag{3.77}$$

This system has its own resonant frequencies.

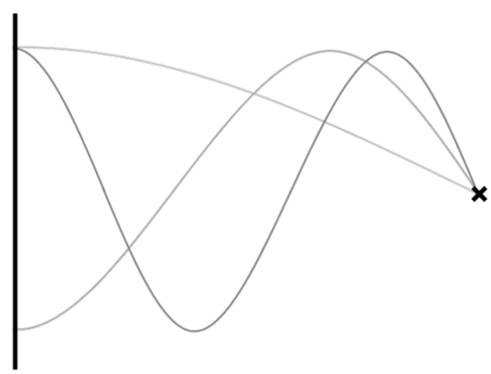

Figure 9: A free and fixed end and the first 3 normal modes.

Now that we understand the conditions in general, this process can be applied to fixed/free ends at any point in space and at any combination (free and free, fixed and free, etc.) but you will always find that the waves fall into these normal modes.

These sorts of problems are ubiquitous in physics. They arise in situations with strict boundary conditions like, for example, quantum mechanics when discussing particles in a box. They are especially applicable to acoustics where the idea of resonance is made much more physically tangible through their connection to musical notes.

3.6 Quick Recap

- For a string with tension T and mass density ρ oscillating at low amplitudes, its height $y(x, t)$ at some point x along it and time t can be characterized by the wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 y}{\partial t^2} \tag{3.78}$$

where the speed $c = \sqrt{\frac{T}{\rho}}$.

- The energy of a string can be broken down into kinetic and potential energy in the following way. Let S be the whole string.

$$K = \int_S \frac{1}{2}\rho \left(\frac{\partial y}{\partial t}\right)^2 \mathrm{d}x \tag{3.79}$$

$$U = \int_S \frac{1}{2}T \left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x \tag{3.80}$$

The total energy is simply $E = K + U$

- For a wave moving in only one direction, $K = U$ which means $E = 2K = 2U$. From this, we know the average energy per unit length of a traveling wave is $\frac{1}{2}\rho A^2\omega^2$.

- Having a boundary condition where there is a sudden change in tension and mass density results in two constraints for the waves on the string: one for continuity and one for vertical force. Let y_L and y_R be the waves to the left and right of the boundary respectively. Assume similar naming for the tensions, mass densities, and speeds. With a boundary at $x = 0$,

$$y_L(0,t) = y_R(0,t) \tag{3.81}$$

$$T_R\left(\frac{\partial y_R}{\partial x}\right) = T_L\left(\frac{\partial y_L}{\partial x}\right) \tag{3.82}$$

- With these constraints, we consider the results of a single incident wave $f_I(x - c_L t)$ approaching the boundary from the left. If $f_R(x + c_L t)$ and $f_T(x - c_R t)$ are the reflected and transmitted waves respectively, then

$$f_R = r f_I = \left(\frac{Z_L - Z_R}{Z_L + Z_R}\right) f_I \tag{3.83}$$

$$f_T = t f_I = \left(\frac{2Z_L}{Z_L + Z_R}\right) f_I \tag{3.84}$$

where r and t are the reflection and transmission coefficients respectively and the **impedance** Z is defined by $Z = \frac{T}{c} = \sqrt{T\rho}$

- Power Flow to be added.

- For constrained end problems, it becomes very useful to use the standing wave picture to constrain certain parts of the string. Fixed ends require $y = 0$ and free ends require $\frac{\partial y}{\partial x} = 0$ at the constrained point.

- When there are two constrained ends, only certain frequencies are allowed on the string. This results in the continuous integral over all possible standing waves to turn into a series sum over allowed frequencies. For the example of two fixed ends at $x = 0$ and $x = L$, the resulting allowed wave numbers and wave are

$$k = \frac{\pi r}{L} \quad \forall r \in \mathbb{N} \tag{3.85}$$

$$y(x,t) = \sum_{r=0}^{\infty} A_r \sin\left(\frac{\pi r}{L}x\right)\cos\left(\frac{\pi r c}{L}t\right) \tag{3.86}$$

Replacing $x = 0$ with a free end results in

$$k = \frac{\pi\left(r + \frac{1}{2}\right)}{L} \quad \forall r \in \mathbb{N} \tag{3.87}$$

$$y(x,t) = \sum_{r=0}^{\infty} A_r \cos\left(\frac{\pi\left(r + \frac{1}{2}\right)}{L}x\right)\cos\left(\frac{\pi\left(r + \frac{1}{2}\right)c}{L}t\right) \tag{3.88}$$

4 Funny Waves (Dispersive)

4.1 Dispersion

Now, we move to more complex waves. Although these waves still follow our standard wave equation, they only partially do. Recall from Section 2.4 that every wave could be expressed as the infinite sum of traveling waves $A\cos(kx \pm \omega t + \phi)$ with $\omega = ck$.

In some media, however, this is not the case. The relation between ω and k may look something like $\omega = c_0 k^2$. This means the speed $c = \frac{\omega}{k} = c_0 k$ which implies that speed is only well defined for single traveling waves. It physically makes sense to think that waves with different frequencies find it easier or harder to move through a certain medium but this is a disaster in terms of the mathematics! The wave equation doesn't even hold because c is not well defined for a general function. This means d'Alembert's solution also doesn't work!

This now makes the traveling waves picture from earlier vitally important. To calculate the motion of any wave, we must decompose it into its constituent traveling waves and then consider the motion of those. Each of this constituent waves will naturally move at different speeds causing the entire wave packet to spread apart and shorten. This is what is referred to as **dispersion** and the media that give rise to it are all called **dispersive media**. Thus far, we have only seen **non-dispersive media**.

4.2 Group vs. Phase Velocity

Now that we do not have a principal velocity c, we must develop more rigorous notions of what velocity means in these media remembering that $\omega = \omega(k)$.

Before moving forward, we define the following useful function.

Definition 3. The phase function is defined as $P(k, x, t, \phi) := kx - \omega(k)t + \phi$.

It simply represents the phase at a certain point in space x and time t with given wave number k and phase offset ϕ. It will help understand the following concepts more intuitively.

We start with the simple case of examining the speed of a single traveling wave.

$$y(x,t) = A\cos(P) \tag{4.1}$$

Let's crack down on what the velocity really represents in this case. We wish to know the speed of the point that has constant height through time. In other words, we may want to know what the speed of the peak is or the speed of the point at phase $\frac{\pi}{2}$. This means the path $x = \sigma(t)$ that describes this constant phase is such that $\frac{\partial P}{\partial t} = 0$.

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(k\sigma - \omega t + \phi\right) \stackrel{!}{=} 0 \tag{4.2}$$

$$\implies k\frac{\mathrm{d}\sigma}{\mathrm{d}t} - \omega = 0 \tag{4.3}$$

$$\implies \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\omega}{k} \tag{4.4}$$

The speed of the path is the familiar $\frac{\omega}{k}$. This is our first type of velocity. Our findings are summarized below.

Definition 4. The **phase velocity** v of a single frequency wave $y = A\cos P$ is defined as the speed of the path such that $\frac{\partial P}{\partial t} = 0$.

Lemma 3.

$$v(k) = \frac{\omega(k)}{k} \tag{4.5}$$

The name is fitting: the velocity of a constant phase.

Now for the slightly more complicated type of velocity: describing the speed of a group of traveling waves. More often than not, our waves do not come in the form of pure sine waves so it is helpful to assign some sort of notion of velocity to these wave forms as well. Let's imagine some wave packet with a peak. From our intuitive knowledge of dispersive media, we know this peak will shorten and spread over time but it will still move in space. We find the velocity of this peak. First, we write our mysterious waveform in the form of traveling waves.

$$y(x,t) = \int_{-\infty}^{\infty} A(k)\cos(P)\mathrm{d}k \tag{4.6}$$

Remember, P depends on k and now $\phi = \phi(k)$.

A peak occurs when multiple waves constructively add up. Let's say there were two waves with wave numbers k_1 and k_2. Say

$$P(k_1, x, t, \phi(k_1)) = P(k_2, x, t, \phi(k_2)) \tag{4.7}$$

These two waves will constructively add up. Alternatively, if they were off by $\pi/2$, for example, they would destroy each other. To find the peak, we must find all the k with similar P so they will constructively add up. We now recall another intuitive fact that waves with similar k will move at approximately the same speed. The k we search for then should be relatively close in value. If they weren't, then the two waves would quickly separate and there would be no peak. This means we are looking for some k_0 around which the phase function of all the waves are approximately the same. To summarize mathematically, we want the path $x = \sigma(t)$ when $\frac{\partial P}{\partial k} = 0$. This yields the following.

$$\frac{\partial P}{\partial k} = \frac{\partial}{\partial k}\left(k\sigma - \omega(k)t + \phi(k)\right) \stackrel{!}{=} 0 \tag{4.8}$$

$$\implies \sigma - \frac{\partial \omega}{\partial k}t + \frac{\partial \phi}{\partial k} = 0 \tag{4.9}$$

$$\implies \frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{\partial \omega}{\partial k} \tag{4.10}$$

Remember, however, that we must choose some value of k_0 to evaluate this derivative at. How do we choose among a whole group of waves? Well, the peak would be where the sum is the highest i.e. where $A(k)$ is the highest. So k_0 is the maximum of $A(k)$ which is the Fourier transform of the wave. Do not worry if you do not understand this, the semantics of which k_0 to choose is not *vitally* important at this stage. Just know that every wave packet has a natural k_0 to choose and, if you know it, you can calculate the speed of the peak using (4.10). This creates our new notion of velocity.

Definition 5. The **group velocity** g of a general wave $y = \int A\cos(P)\mathrm{d}k$ is defined as the speed of the path such that $\frac{\partial P}{\partial k} = 0$.

Lemma 4.

$$g = \frac{\partial \omega}{\partial k}(k_0) \tag{4.11}$$

where k_0 is the point where $A(k)$ has a maximum.

Note that this definition is only practically useful when looking at single wave packets with a single well-defined peak. Although you can get a numerical group velocity for any arbitrary wave, it does not serve the motivation for its creation.
We have now made ourselves once again comfortable with the idea of velocity in this funny media.

4.3 Physical Interpretation

Often, the group velocity is *more* important than the wave velocity although it seems like a more contrived definition. This is because this is the rate at which information is transferred. Say you wish to send a signal through a dispersive media. Unless you can instantaneously generate a sine wave across all space, it cannot yield the phase velocity. Your wave will always start as some small packet that immediately disperses traveling at group velocity. We must move away from the comfort of classical notions of velocity to this more practical definition.

4.4 Quick Recap

- There exist media such that the speed of a wave is not well defined for arbitrary waves i.e. $\omega \neq ck$. Instead they adopt more complex relations such as $\omega = c_0 k^2$. These are called **dispersive** media because wave packets tend to go through **dispersion** over time as their constituent waves spread by traveling at different velocities.
- The phase function $P(k, x, t, \phi) := kx - \omega(k)t + \phi$
- We define the **phase velocity** v of a single frequency wave $y = A\cos(P)$ as the speed of the path where there is constant phase i.e. $\frac{\partial P}{\partial t} = 0$. This results in the relation

$$v(k) = \frac{\omega(k)}{k} \tag{4.12}$$

- We define the **group velocity** g of a general wave $y = \int A\cos(P)\mathrm{d}k$ as the speed of the peak in a given a wave packet or, more rigorously, where $\frac{\partial P}{\partial k} = 0$. This results in the relation

$$g = \frac{\partial \omega}{\partial k}(k_0) \tag{4.13}$$

  where k_0 is the frequency at which $A(k)$ is maximal
- Group velocity is often more practically useful as it gives the rate of information transfer