C Flows

It's very common in physics to talk about flow which, primitively speaking, is simply things moving around in space like a fluid. The first rigorous instance of this for an undergraduate will most likely be in the kinetic theory of gases where one treats particles as a fluid by looking at particle and momentum densities or temperature flows. However, the same ideas also creep into electromagnetism in subtle ways with charge densities forming currents and later more intricately with electromagnetic radiation. They become even more relevant when talking about actual fluid flows or, in more abstract settings, flows in Hamiltonian phase space. We even see it with quantum probability amplitudes! Needless to say, flow is an ubiquitous concept and its familiarity is an important tool for a physicist. In this appendix, we develop a nice simple formalism that helps us think about these flows.

The first way to describe a property's flow in space is to look at its density in space and time $\rho(\mathbf{r}, t)$ [7] This could be the number of particles, energy, mass, etc. For now, we will refer to whatever this quantity is as "the substance". By definition, we say the amount of substance S in some volume $\mathcal{V}$ at time t is

$$S(\mathcal{V}, t) = \int_{\mathcal{V}} \rho(\mathbf{r}, t) \mathrm{d}V \tag{C.1}$$

Now consider an alternative measurement of the substance. Consider setting up a wall and tracking the rate at which this property flows through it. For energy, for example, it would be the power output to the wall. We could then rotate this patch in all directions to get a vector rate for the flow. This quantity is clearly a well-defined and intuitive notion of how the property is moving in space and time. If we take this flow per area per time, we call this the **flux density** and denote it $\mathbf{J}(\mathbf{r}, t)$. Now, say we had some volume V and wanted to know how much substance was leaving it at any second. Well, by definition, any patch $\mathrm{d}\mathbf{A}$ lets out $\mathbf{J} \cdot \mathrm{d}\mathbf{A}\mathrm{d}t$ at any given moment. This means the rate at which the volume is *losing* the substance from flow is

$$-\frac{\mathrm{d}S(\mathcal{V}, t)}{\mathrm{d}t} = \int_{\partial\mathcal{V}} \mathbf{J}(\mathbf{r}, t) \cdot \mathrm{d}\mathbf{A} \tag{C.2}$$

Note: it is the amount that the volume loses and not gains by the way we define $\mathrm{d}\mathbf{A}$ which is naturally an outward vector by convention. This is the source of the negative on the left.

The connection between the density and flux follow naturally

$$\int_{\mathcal{V}} \frac{\partial \rho}{\partial t} \mathrm{d}V = \frac{\mathrm{d}S(\mathcal{V}, t)}{\mathrm{d}t} = -\int_{\partial\mathcal{V}} \mathbf{J} \cdot \mathrm{d}\mathbf{A} \tag{C.3}$$

Here, there is no better application of Gauss' law.

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t} \tag{C.4}$$

This is called the **continuity equation**. We took two really intuitive concepts: density and flux and compactly put them together in one equation. You may however notice a few uncomfortable things. Namely, the flux is not a unique vector given a density. This, however, makes perfect sense. Imagine adding a constant vector to $\mathbf{J}$ which would not have any effect on the charge density. Physically, this would be like adding the same flux everywhere in space at all times. We would never be able to see the effect of this

[7] For now, we will treat this is a physical space for intuition but this can extend to phase spaces as well of course.

because the same amount enters a point as exits. We would only sense this difference from measuring with a wall as described earlier. What it does however tell us is that if we have a vector and scalar field that satisfy the properties above, then the vector can be thought of as a flux for the scalar in a physical sense.

C.1 Mean Flow

Let us consider a fairly simply flow example which will prove instructive for the future.

Consider a distribution of non-interacting particles in space all moving with velocity $\mathbf{v}$. We also assume no particles are created/destroyed. Let $n(\mathbf{r}, t)$ represent particle density. Of course, this function cannot have any form. If the velocity is in the positive x direction, this function should looks like it's moving in the positive x direction. So let's put some limitations on it.

Say we have some amount of particles at $(\mathbf{r}, t)$. We know a time Δt ago, they must have been at position $\mathbf{r} - \mathbf{v}\Delta t$. This puts the following restriction on n.

$$n(\mathbf{r}, t) = n(\mathbf{r} - \mathbf{v}\Delta t, t - \Delta t) \tag{C.5}$$

We can then rearrange a little.

$$\frac{n(\mathbf{r}, t) - n(\mathbf{r} - \mathbf{v}\Delta t, t - \Delta t)}{\Delta t} = 0 \tag{C.6}$$

And now take a limit because it holds for all time intervals.

$$\lim_{\Delta t \to 0} \frac{n(\mathbf{r}, t) - n(\mathbf{r} - \mathbf{v}\Delta t, t - \Delta t)}{\Delta t} = \mathbf{v} \cdot \nabla_{\mathbf{r}} n + \frac{\partial n}{\partial t} = 0 \tag{C.7}$$

Through a simple vector identity, we can show that $\mathbf{v} \cdot \nabla n = \nabla \cdot (n\mathbf{v})$ and the following result is achieved.

$$\nabla \cdot (n\mathbf{v}) = -\frac{\partial n}{\partial t} \tag{C.8}$$

This is a beautifully simple result. The flux for n is simply $n\mathbf{v}$. We use this result to motivate a very general concept known as the **mean flow**. For some property density ρ with flux density $\mathbf{J}$, the mean flow is the vector that satisfies $\mathbf{J} = \rho\mathbf{u}$. In general, this vector can be thought of as the "velocity" of the property at a given point in space and time which we know is quite literal in the simple case we just went through. It makes intuitive sense, however, as we can see it always has dimensions of velocity regardless of the property and it implies the flux density is simply how many particles are at a point times how fast they are moving, also intuitive.

If we were to remove the restriction that all particles are moving at the same velocity, then the mean flow vector would be the *average* velocity at a point (exercise for the reader. Hint: flux vectors are additive). If we were to add other ideas like collisions, forces, etc., then this mean flow would represent some *effective* average velocity. Regardless, it is a source of great physical insight.

C.2 Breaking Conservation: Sources and Sinks

In some cases, conservation of a property does not hold. Say, for the sake of argument, that at every point in space particles are destroyed at a rate of λ per unit volume per time. Then, in fact, we would need the following addition.

$$\frac{\partial n}{\partial t} = -\nabla \cdot (n\mathbf{u}) - \lambda \tag{C.9}$$

Based on if the property is increasing or decreasing, this is called a **source** or a **sink**. In this case, it is a sink.

In fact, any property is subject to this breaking of conservation. Ignore particles now and think about any substance in space. Say, for example, we have electromagnetic energy u traveling in space and there is a charged particle. It will absorb some energy and convert it into kinetic energy which would be a sink. Maybe a light is turned on with power density $P(\mathbf{r})$ and we would have the equation

$$\frac{\partial u}{\partial t} = -\nabla \cdot \mathbf{J}_u + P \tag{C.10}$$

The reason our very first relation between the density and flux now fails is because when we considered $\frac{\mathrm{d}S}{\mathrm{d}t}$, we only considered the change due to *flow* assuming that the substance was already conserved. In fact, say you are given a vector $\mathbf{J}$ and ρ that are known to be, in some sense, a flux and density for a substance, we then say the substance is **locally conserved** if it follows (C.4). The continuity equation is then not really a strict relation between flux and density but an assertion of conservation.

C.3 Convective Derivative

Consider now an observer inside the flow moving along with the mean flow $\mathbf{u}$ and, at each point, tracking the property density. Let the observer's position be denoted by $\mathbf{x}(t)$ and the density he measures be $\overline{\rho}(t) = \rho(\mathbf{x}(t), t)$. How does this quantity change?

$$\frac{\mathrm{d}\overline{\rho}}{\mathrm{d}t} = \frac{\partial \rho}{\partial t} + \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} \cdot \nabla_{\mathbf{x}}\rho \tag{C.11}$$

$$= \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho \tag{C.12}$$

From this, we motivate the **convective derivative** defined as

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} := \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho \tag{C.13}$$

which tells us exactly how this observer observes the change in the property. This observer is usually seen as some fluid element which we know, by definition, moves with the mean flow velocity.

This motivates a common property of flows. We call a flow **incompressible** if $\frac{\mathrm{D}\rho}{\mathrm{D}t} = 0$. To understand the naming, consider again the particle distribution. Select some volume of particles and color them say purple to better visualize them. The purple blob will initially take up some volume. If the convective derivative is zero, then the density of purple particles never change. We know, by definition, the number of particles particles don't change. This must mean the total volume of the purple must not change i.e. it is incompressible (and inexpandable). In general, this means the volume of a fluid element never changes, a very important concept in many contexts!

The definition of the convective derivative however is quite clunky and it may become hard to check this condition so consider the following. Assume we are working with a conserved property. Then we note that

$$\begin{aligned} 0 &= \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) && \text{(C.14)} \\ &= \frac{\partial \rho}{\partial t} + \mathbf{u} \cdot \nabla \rho + \rho(\nabla \cdot \mathbf{u}) && \text{(C.15)} \\ &= \frac{\mathrm{D}\rho}{\mathrm{D}t} + \rho(\nabla \cdot \mathbf{u}) && \text{(C.16)} \end{aligned}$$

This implies

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho(\nabla \cdot \mathbf{u}) \tag{C.17}$$

We then see that, under conserved flow, an equivalent condition for incompressibility is $\nabla \cdot \mathbf{u} = 0$.