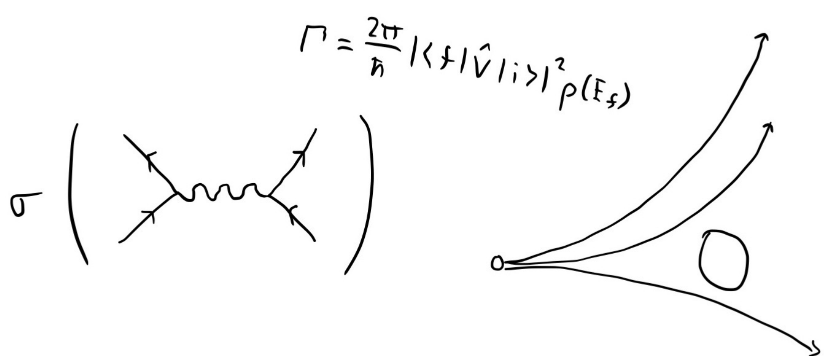
Scattering Theory



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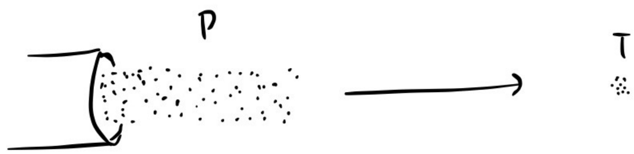
1 Introduction

The main vehicle for understanding the structures of different materials and/or properties of different microscopic entities is through **scattering**. Scattering is essentially shooting particles at these materials/entities and looking at the outcome to deduce any properties we can about the system. A very charged target for example will deflect a lot, different potentials will deflect in different patterns, etc. Here, we will learn the specifics of how to do this.

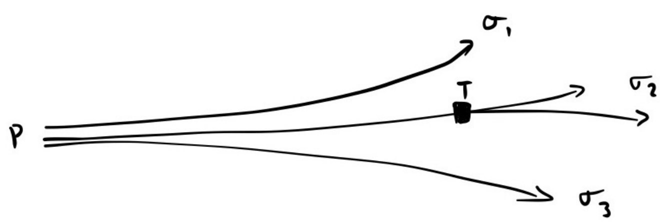
2 Cross-Section

2.1 Definition

Say we use as targets particles called T and projectiles particles called P.[1] We continuously fire P with number current density j at some collection of T particles.[2] We assume that each P interacts with just one T. These collisions are usually called **scattering events**.



Scattering can encompass many different possible events i.e. P can deflect elastically or inelastically, P can get sucked in by T and then re-emitted later or not, T and P can react and form any amount of new particles that go in different directions, etc. Let's call the space of all these events Σ denoting an event as $\sigma \in \Sigma$.



Now say we choose some set of events $S \subset \Sigma$. We want to create some measure how how often these events occurs relative to the other events. One thing we could do is simply calculate the rate of these events in this set S occurring i.e. the number per time. Call this $\dot{N}(S)$. We notice, however, that scaling the current density j scales this value by the same amount i.e. firing twice as many particles results in twice as many events happening. The same is true for the number of T particles N_T because the more scattering targets, the more of the beam that is scattered. We don't really want a property of a scattering event to depend on how many particles were sent or hit. We want it to only depend on the interaction of P and T. We could then consider instead $\dot{N}/(jN_T)$. This is called the **cross section**.

(Fix the notation: events and cross section are notated the same)

Definition 1. The **cross section** of some set of events $S \subset \Sigma$ is defined as

$$\sigma(S) = \frac{\dot{N}(S)}{jN_T} \tag{2.1}$$

where $N(S)$ is the number of times an event in S happens per unit time, N_T the number of targets, and j

[1] We use particles to refer to anything from actual fundamental particles to atoms to molecules to simply arbitrary objects.

[2] Number current density $\mathbf{j}$ is the value such that $\mathbf{j} \cdot d\mathbf{A}dt$ is the number of particles passing through an area patch $d\mathbf{A}$ in time dt. If particle number is conserved, this obeys the equation $\nabla \cdot \mathbf{J} = -\frac{\partial n}{\partial t}$ where n is particle number density

the number current density of the projectiles.

Alternatively, we may say that that $N_T = \eta_T A$ where η_T is the area number density of targets and A is the cross sectional area of the beam. We also know $j = I/A$ where I is the number current of the incoming particles. As a result, we can also say

$$\sigma = \frac{\dot{N}}{N_T j} = \frac{\dot{N}}{\eta_T I} \tag{2.2}$$

Note that cross section has units of area. We can interpret this as the following. Consider, for example, some set of particles P passing through a patch of area A going at one target i.e. $N_T = 1$. The rate of particles passing through, by definition, is jA. If we consider these to be the particles to eventually scatter into the set of events S, then we have that $\sigma(S) = (jA)/j = A$. In other words, the cross section is how much cross sectional area of the initial beam a single T particle was able to scatter into the set S.

Of course, this interpretation falls apart, for example, when we consider events that cause particles to split (σ_2 in the image above) or in quantum mechanics where the same initial particles decay into multiple final states. However, even in those settings, it is nonetheless true that cross section provides a good metric for the amount of scattering that has occurred.

It should also be noted that both $\dot{N}$ and j are relatively easy quantities to measure. $\dot{N}$ merely involves counting the amount of some result and j we can set when creating the experiment. As a result, cross section is a natural measurement in experiment.

2.2 Partitioning

Proposition 1. Consider a countable set of disjoint event sets $\{S_i\}$ i.e. $S_i \cap S_j = \varnothing$ for $i \neq j$. This means

$$\sigma\left(\bigcup_i S_i\right) = \sum_i \sigma(S_i) \tag{2.3}$$

The proof is trivial and left to the reader.[3] All this proposition says is that we can add the cross sections of different event sets so as long as there is no overlap.

A common partition of the event space is elastic, inelastic, and absorption. Because these properties of scattering are mutually exclusive i.e. disjoint, we can conclude, from above, that

$$\sigma = \sigma_{\text{elastic}} + \sigma_{\text{inelastic}} + \sigma_{\text{absorption}} \tag{2.4}$$

Inelastic here usually just means not elastic or absorbed.

2.3 Differential cross section

Often times, we will also see that the event space can be reduced into the form $\Sigma = S^2 \times A$. This essentially means that each event will be defined by two things: the outgoing direction encoded in S^2 and the event type

[3]It is clear that the statements $\sigma(\varnothing) = 0$ and $\sigma(S) \geqslant 0$ are true. Combining them with the proposition above means that cross section σ is a measure over the event space Σ.

encoded in A.[4] For example, consider dihydrogen being shot at oxygen. Say there are only two distinct types of events: the dihydrogen simply deflects off the oxygen by Coulomb interaction, the dihydrogen combines with oxygen to form H_2O. Call these two types A_1 and A_2. Each event can also further be classified into which direction the dihydrogen (A_1) or H_2O (A_2) goes in. Each direction corresponds to some point on the unit sphere S^2. This means the event space is $\Sigma = S^2 \times \{A_1, A_2\}$.

We then define that **differential cross section** $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ as the amount of cross section per solid angle or

Definition 2. The **differential cross section** $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ is the function such that

$$\sigma(S) = \int_{S \subset S^2} \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \mathrm{d}\Omega \tag{2.5}$$

A patch of solid angle on the sphere $\mathrm{d}\Omega$ has cross section $\mathrm{d}\sigma = \frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}\mathrm{d}\Omega$. The sphere S^2 can refer to all the directions for a specific event type A_i, several event types, or all of them.

If, for example, you knew the particle rate per solid angle $\dot{n}(\theta, \phi)$ i.e.

$$\dot{N}(S) = \int_S \dot{n} \mathrm{d}\Omega \tag{2.6}$$

Then, we can easily show that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{jN_T} \frac{\mathrm{d}\dot{N}}{\mathrm{d}\Omega} = \frac{\dot{n}}{jN_T} \tag{2.7}$$

2.4 Summary

- We consider colliding a stream of particles P at a set of targets T. This is called **scattering**. We call the set of all possible scattering outcomes Σ
- We can construct a measure of how much T scatters P into some subset of event space $S \subset \Sigma$ using the **cross section** σ.

$$\sigma(S) = \frac{\dot{N}(S)}{jN_T} \tag{2.8}$$

  where $\dot{N}(S)$ is number of times an event in S occurs per unit time, N_T is the number of targets, and j is the number current density of P.
- The cross section of S, in cases where no particles are created/destroyed and we stay in the classical regime, can be interpreted as the amount of cross-sectional area a single T was able to scatter the Ps into an event in S.
- The cross section for countable mutually disjoint sets $\{S_i\}$ obeys

$$\sigma\left(\bigcup_i S_i\right) = \sum_i \sigma(S_i) \tag{2.9}$$

[4] $\times$ denotes the Cartesian product of the two sets i.e. the set of all pairs with one from each

As a result, a common partition of the cross section is

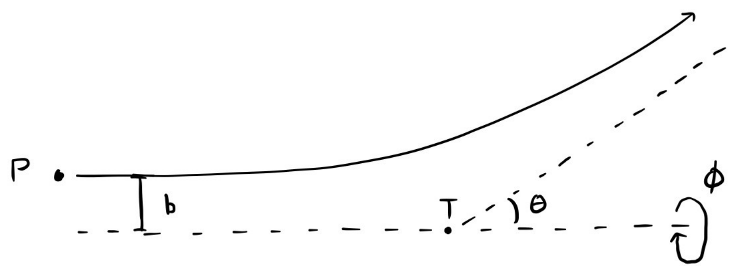
$$\sigma = \sigma_{\text{elastic}} + \sigma_{\text{inelastic}} + \sigma_{\text{absorption}} \tag{2.10}$$

- We can often split the event space into $\Sigma = S^2 \times A$ where S^2 gives the final direction and A gives the event type. As a result, we usually define **differential cross section** $\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega}$ as the amount of cross section per solid angle.

3 Classical Scattering Problems

We consider 2 simple cases of scattering in the classical regime where P approaches T which has some central potential and is simply deflected away in all cases. It is clear that the event space is $\Sigma = S^2$ i.e. we label events by the final direction of the particle's momentum. Our goal in each case is to arrive at an expression for the differential cross section. We can streamline this process in the following way.

All of the cases will have the same general structure as the image below.



The impact parameter is b and we parametrize the outgoing momentum sphere with the regular spherical coordinates (θ, ϕ) with $\theta = 0$ corresponding to no deflection. For each incoming (b, ϕ), there exists an outgoing (θ, ϕ). Consider the particles $\mathrm{d}\dot{N}$ hitting at the specific solid angle corresponding to (θ, ϕ). It will be $\mathrm{d}\dot{N} = \dot{n}\mathrm{d}\Omega$ for some $\dot{n}$. The same particles will correspond to some (b, ϕ) in the incoming beam where $\mathrm{d}\dot{N} = j\mathrm{d}A = jb\mathrm{d}b\mathrm{d}\phi$. Setting these two equal, we get that

$$\dot{n} = \frac{jb\,\mathrm{d}b\,\mathrm{d}\phi}{\sin\theta\mathrm{d}\theta\mathrm{d}\phi} = \frac{jb}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{3.1}$$

The absolute value comes from the fact that what we are actually calculating here is a Jacobian.[5]. Remembering that $N_T = 1$ here, the differential cross section from (2.7) should be

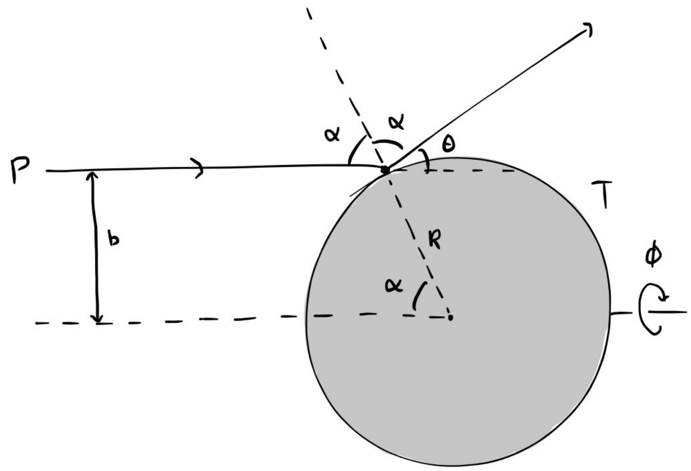
$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\dot{n}}{j} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{3.2}$$

This means that all we need to do to get the differential cross section is calculate the relationship between b and θ. This is intuitively obvious but now we have a proper expression.

3.1 Hard Ball

Consider scattering off a hard ball of radius R. The following image then represents the situation at hand.

[5] In actuality, we should be calculating $\dot{n} = \frac{jb}{\sin\theta}\left|\frac{\partial(b,\phi)}{\partial(\theta,\phi)}\right|$ but ϕ neither changes nor affects the relationship between b and θ in general so what we do here is acceptable. Be wary though.



By geometry, we can deduce that $\theta + 2\alpha = \pi$ so

$$\alpha = \frac{\pi - \theta}{2} \tag{3.3}$$

Noting that $\sin\alpha = \frac{b}{R}$, we get that

$$\cos\left(\frac{\theta}{2}\right) = \frac{b}{R} \tag{3.4}$$

From here, we use (3.2) to find $\frac{d\sigma}{d\Omega}$.

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right| \tag{3.5}$$

$$= \frac{1}{\sin\theta}\left(R\cos\frac{\theta}{2}\right)\left(\frac{R}{2}\sin\frac{\theta}{2}\right) \tag{3.6}$$

$$= \frac{R^2}{4} \tag{3.7}$$

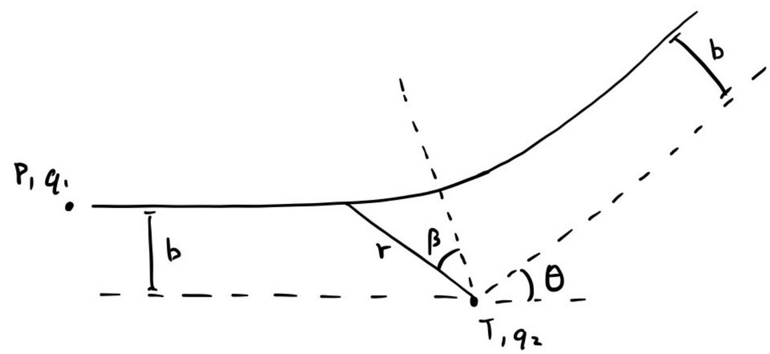
This is our final result! We can see clearly that the total cross section is the result of integrating this over the sphere which simply yields

$$\sigma = \pi R^2 \tag{3.8}$$

This makes sense as the total scattered cross sectional area is the same as the actual cross sectional area of T.

3.2 Point Charge

We now consider deflection of a point charge. This is usually called **Rutherford scattering** as it involves Coulomb deflection without any other effects like excitation of the atoms. We then have the following setup.



Note the addition of two new variables r, the distance from the target, and β the angle with respect to the line from the target to midway point. Call the direction this line indicates the $\overline{y}$ direction. It is also important to notice that the particle also leaves at distance of b from the line extending out of the target in the deflection direction. This is a basic result of the conservation of angular momentum.

First, let us state the conservation of angular momentum L. Initially, it has $L = mv_0 b$ and, at some arbitrary point, has $L = mr^2\dot{\beta}$. Here, v_0 is the initial velocity. Setting the two equal yields

$$\mathrm{d}t = \frac{r^2}{v_0 b}\mathrm{d}\beta \tag{3.9}$$

Let's now calculate the momentum change in the $\overline{y}$ direction Δp. We can deduce from geometry that

$$\Delta p = 2p_{\overline{y}0} = 2mv_0 \sin\theta/2 \tag{3.10}$$

We calculate the same quantity considering the force in that direction.

$$\Delta p = \int_{-\infty}^{\infty} F\cos\beta\,\mathrm{d}t \tag{3.11}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\cos\beta}{r^2}\,\mathrm{d}t \tag{3.12}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 v_0 b}\int_{-(\frac{\pi}{2}-\frac{\theta}{2})}^{\frac{\pi}{2}-\frac{\theta}{2}}\cos\beta\dot{\beta} \tag{3.13}$$

$$= \frac{q_1 q_2}{4\pi\epsilon_0 v_0 b}\sin(\beta)|_{-(\frac{\pi}{2}-\frac{\theta}{2})}^{\frac{\pi}{2}-\frac{\theta}{2}} \tag{3.14}$$

$$= \frac{q_1 q_2}{2\pi\epsilon_0 v_0 b}\cos\left(\frac{\theta}{2}\right) \tag{3.15}$$

where we have plugged in (3.9) from earlier. Equating the two expressions for Δp, we can deduce that

$$b = \frac{q_1 q_2}{4\pi\epsilon_0 m v_0^2} \cot\left(\frac{\theta}{2}\right) \tag{3.16}$$

From here, we can calculate cross section as usual.

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{q_1 q_2}{4\pi\epsilon_0 m v_0^2}\right)^2 \frac{1}{2}\frac{\cot(\theta/2)\csc^2(\theta/2)}{\sin\theta} \tag{3.17}$$

$$= \left(\frac{q_1 q_2}{8\pi\epsilon_0 m v_0^2}\right)^2 \frac{1}{\sin^4(\theta/2)} \tag{3.18}$$

This is called the **Rutherford scattering formula**.

We can also rewrite it by considering the distance of closest approach d. This will occur when the kinetic energy has completely converted to potential energy i.e. when

$$\frac{m v_0^2}{2} = \frac{q_1 q_2}{4\pi\epsilon_0 d} \tag{3.19}$$

$$\implies d = \frac{q_1 q_2}{2\pi\epsilon_0 m v_0^2} \tag{3.20}$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{d^2}{16\sin^4(\theta/2)} \tag{3.21}$$

If we integrate this over S^2, we find that the total cross section for all possibilities actually diverges. This is clear from the fact that Coulomb force has infinite range so it still properly scatters particles at arbitrarily large impact parameters. It turns out, however, that in quantum mechanics, you can have finite cross section even for infinite range forces and we will see this soon.

3.3 Summary

- For a classical scattering problem where we only see deflection from a central potential i.e. $\Sigma = S^2$, it is the case that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{b}{\sin\theta}\left|\frac{\mathrm{d}b}{\mathrm{d}\theta}\right| \tag{3.22}$$

  where b is the impact parameter and θ is the deflection angle.

- For a hard ball of radius R, we get that

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{R^2}{4} \tag{3.23}$$

$$\sigma(\Sigma) = \pi R^2 \tag{3.24}$$

- For a point charge where P and T have charges q_1 and q_2 respectively,

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \left(\frac{q_1 q_2}{8\pi\epsilon_0 m v_0^2}\right)^2 \frac{1}{\sin^4(\theta/2)} = \frac{d^2}{16\sin^4(\theta/2)} \tag{3.25}$$

$$\sigma(\Sigma) \to \infty \tag{3.26}$$

where d is the distance of closest approach. (3.25) is known as the **Rutherford scattering formula** and this type of scattering is known as **Rutherford scattering**.

- Infinite range potentials will result in infinite cross section in classical settings.

4 Quantum Scattering Problems

We now move to the true home of scattering theory: quantum mechanics. For this, we generally consider some incoming particle in some specific eigenstate $|E\rangle$ interacting with another particle and leaving with some wavefunction. The way we usually approach this problem is by quantifying the rate at which some incoming state $|E_i\rangle$ transitions into a given outgoing state $|E_f\rangle$ as a result of this potential. Knowing all these rates gives us an idea of how the target particle scattered the incoming one and, in the end, will give us our desired differential cross section.

4.1 Perturbation Theory

Let's consider the general situation. We have some initial state $|\Psi\rangle$. This is an eigenstate of some Hamiltonian $\hat{H}_0$. Say we model the scattering event as the turning on of some Hamiltonian $\Delta H(t)$ so that $\hat{H} = \hat{H}_0 + \Delta H(t)$. Consider the eigenvectors $|n\rangle$ and eigenenergies E_n of H_0. We can then write the general solution, without assuming anything, as

$$|\Psi\rangle = \sum_n c_n(t) e^{-iE_n t/\hbar} |n\rangle \tag{4.1}$$

Applying the full Schrodinger equation and taking the nth component $\langle n|i\hbar\partial_t|\Psi\rangle = \langle n|H|\Psi\rangle$ yields the result

$$\dot{c}_n = \frac{1}{i\hbar} \sum_k c_k(t) e^{-i(E_k - E_n)t/\hbar} \langle n|\Delta H_0|k\rangle \tag{4.2}$$

We clearly have no general analytic solution to this at the moment. Here is where we invoke **perturbation theory**. Consider the scattering potential to be some continuous perturbation on the underlying Hamiltonian H_0 i.e.

$$H(t;\lambda) = H_0 + \lambda \Delta H(t) \tag{4.3}$$

We then propose solutions of the form

$$|\Psi(t;\lambda)\rangle = \sum_{k \in \mathbb{N}_0} \lambda^k |\Psi_k(t)\rangle \tag{4.4}$$

We call $|\Psi_k\rangle$ the kth order term like in a Taylor series. Looking at the coefficients $c_n^{(k)} = \langle n|\Psi_k\rangle$, we get that

$$c_n = \sum_k \lambda^k c_n^{(k)} = c_n^{(0)} + \lambda c_n^{(1)} + \ldots \tag{4.5}$$

Plugging this into (4.2), we get that

$$\sum_p \lambda^p \dot{c}_n^{(p)} = \frac{1}{i\hbar} \sum_p \sum_k \lambda^{p+1} c_k^{(p)} e^{-i(E_k - E_n)t/\hbar} \langle n|\Delta H_0|k\rangle \tag{4.6}$$

where the extra λ on the RHS comes from the fact that $\Delta H_0 \to \lambda \Delta H_0$. If we take this to be true for all λ, then by matching orders, we get that

$$\dot{c}_n^{(0)} = 0 \tag{4.7}$$

$$\dot{c}_n^{(p)} = \frac{1}{i\hbar} \sum_k c_k^{(p-1)} e^{-i(E_k - E_n)t/\hbar} \langle n|\Delta H_0|k\rangle \tag{4.8}$$

We know then that the zeroth order is constant. What is this constant value? If we revisit (4.5) and require it to be true at $t = 0$ and for all λ, then we get that

$$c_n^{(0)} = c_n(t = 0) \tag{4.9}$$

If we assume that the incoming particle starts in some initial eigenstate $|i\rangle$, then $c_n^{(0)} = \delta_{in}$. (FIX THIS SECTION UP WITH INITIAL CONDITIONS AND SUCH AND EXPLANTAION OF WHY HIGHER ORDERS ARE EXPECTED TO DECAY, CHANGE MATRIX ELEMENT NOTATION)

As expected, to zeroth order, the wavefunction does not change. The next few sections however will be devoted to understanding the higher order terms.

4.2 First-Order Transitions

Let's apply (4.8) for the first time.

$$\dot{c}_n^{(1)} = \frac{1}{i\hbar} \sum_k c_k^{(0)} e^{-i(E_k - E_n)t/\hbar} \langle n|\Delta H_0|k\rangle = \frac{1}{i\hbar} e^{-i(E_i - E_n)t/\hbar} \langle n|\Delta H_0|i\rangle \tag{4.10}$$

applying our zeroth order result from last section.

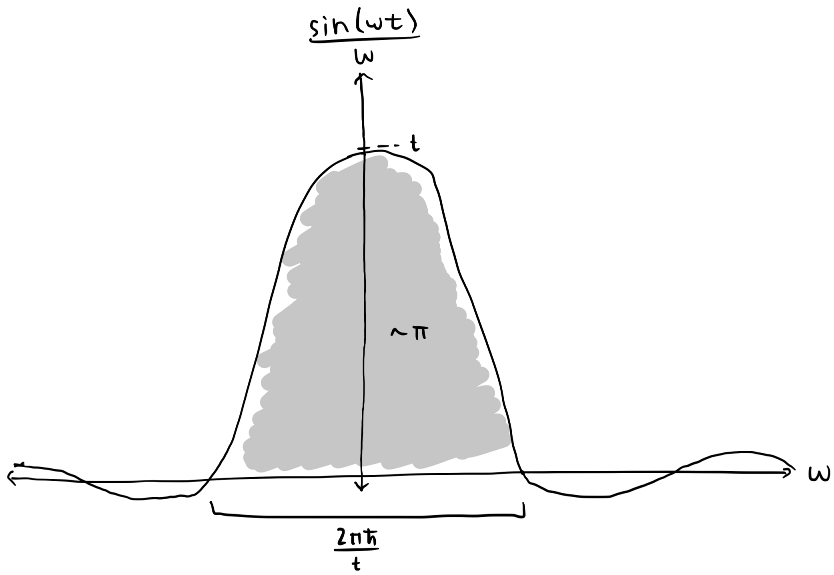
We now wish to calculate the rate at which each final state $|f\rangle$ gains probability as this will give us the particle rate for a given event and, in the end, a cross section. The quantity we want to calculate then is

$$\Gamma_{i\to f} = \frac{\mathrm{d}|c_f|^2}{\mathrm{d}t} \tag{4.11}$$

Using our results above, we simply carry out the algebra assuming our matrix element is constant up to this order

$$\begin{aligned}
\Gamma_{i\to f} &= 2\mathrm{Re}\{\dot{c_f}^* c_f\} && (4.12)\\
&= 2\mathrm{Re}\left\{-\frac{1}{i\hbar}e^{i(E_i-E_n)t/\hbar}\mathcal{M}_{in}\cdot\int_0^t \frac{1}{i\hbar}e^{-i(E_i-E_n)t'/\hbar}\mathcal{M}_{ni}\mathrm{d}t'\right\} && (4.13)\\
&= \frac{2|\mathcal{M}_{in}|^2}{\hbar^2}\mathrm{Re}\left\{e^{i(E_i-E_n)t/\hbar}\int_0^t e^{-i(E_i-E_n)t'/\hbar}\mathrm{d}t'\right\} && (4.14)\\
&= \frac{2|\mathcal{M}_{in}|^2}{\hbar^2}\mathrm{Re}\left\{\frac{\hbar}{-i(E_i-E_n)}e^{i(E_i-E_n)t/\hbar}e^{-i(E_i-E_n)t'/\hbar}\Big|_0^t\right\} && (4.15)\\
&= \frac{2|\mathcal{M}_{in}|^2}{\hbar^2}\frac{\sin\omega t}{\omega} \quad \text{where } \omega := \frac{E_f-E_i}{\hbar} && (4.16)\\
& && (4.17)
\end{aligned}$$

We now make one further approximation. Consider the graph of $\frac{\sin\omega t}{\omega}$ (FIX GRAPHIC WITH HBAR)



We can see that as time goes on, this curve becomes sharply peaked around $\omega = 0$ but retains an area of $\approx \pi\hbar$ underneath. As a result, we just say $\sin(\omega t)/\omega \approx \pi\hbar\delta(\omega)$ where the delta function is 1 at $\omega = 0$, not infinite. We then get the final rate to be

$$\Gamma_{i\to f} = \frac{2\pi}{\hbar}|\mathcal{M}_{if}|^2\delta(E_f - E_i) \tag{4.18}$$

This is called **Fermi's Golden Rule**. It says that the rate at which some initial state evolves into some final state is proportional to the associated matrix element squared. The delta function on the end is the expression of conservation of energy: you can't transition into a state of different energy.

Let's calculate now the differential cross section of a quantum scattering event to first order. Consider some incoming particle state $|\mathbf{k}_i\rangle$ where

$$\langle \mathbf{r}|\mathbf{k}\rangle = \frac{1}{\sqrt{V}}e^{-i\mathbf{k}\cdot\mathbf{r}} \tag{4.19}$$

where V is the volume of the system these particles are in. This quantity will soon disappear. Say N particles are coming in per unit time. Then the amount of particles that transition into some final direction $\mathbf{k}$ per unit time $\mathrm{d}\dot{N}$ is

$$\mathrm{d}\dot{N} = N\Gamma_{i\to f}\rho(\mathbf{k})\mathrm{d}^3\mathbf{k} = N\Gamma_{i\to f}\frac{V}{(2\pi)^3}\mathrm{d}^3\mathbf{k} \tag{4.20}$$

where ρ is the state density. We know from energy conservation though that $k = k_i$ so our state space is really a sphere of radius k_i.

4.3 Second-Order Transitions (Resonances)

We saw, in the last section, that the first order wavefunction can be interpreted as various simultaneous direct transitions from the initial state to any given final state. We will now see that second order admits a similar but fairly distinct interpretation and will help us understand the concept of resonance.

4.3.1 Resonances

4.4 General Transitions (Feynmann Diagrams)

A Natural Units

It turns out that in the most fundamental theories of the universe, some values always come along with some constant. For example, the magnetic field B always comes as B/c, time t as ct, action S as $S/\hbar$, temperature T as k_BT, etc. Let's think about the t case for now.

Because in all fundamental equations, $t' = ct$ is what appears, t' must be the physically important quantity. The universe only knows t' and $t = t'/c$ is simply that scaled by some random constant $c \approx 3 \cdot 10^9 m/s$. We intuitively however only understand t which is measured in seconds, something we understand, as opposed to t' which is measured in meters. How do we reconcile the fact that nature and our intuition disagree on which to focus on? This can be done using **natural units**.

Natural units proposes that these constants are simply unit conversions i.e. that $t' = t$. This must mean that $c = 1$ or that

$$3 \cdot 10^8\,\mathrm{m} = 1\,\mathrm{s} \tag{A.1}$$

$$1\,\mathrm{m} = 3 \cdot 10^{-9}\,\mathrm{s} \tag{A.2}$$

This means that 1 meter becomes approximately 3 nanoseconds. Note, however, that we did not change the value of c but merely added an additional constraint onto it. This means you can drop the c and still work in conventional units as normal writing expressions like $\mathrm{m/s^2}$ but having the ability to, at any point, convert it to a physically relevant quantity using the unit conversions above. Natural units allow for a seamless unity of what is physically and intuitively relevant by merely interpreting meters as some scaled seconds.

We should note that this is not just a matter of convenience but is a profoundly physical thing to say that the coefficient c is merely a unit conversion. Time and space *should* be able to be measured as the same thing as in relativity we add them together all the time. The impracticality of measuring time in meters and space in seconds though comes from the fact that they have much different magnitudes and for everyday applications, meters are the best for space and seconds the best for time on the scales we see them. On astronomical scales however, we use it all the time like, for example, when we measure things in light-years. In natural units, a light year is just a year and it makes perfect sense to measure distances in that way. Note here, that saying that a distance measured in time does have physical meaning. Something that is 1 second away in space can only affect me 1 second from now in time. The connection between the measurement of something in conventional space and time units is very real.

With that all being said, I present the following description of what are called **Planck units**, often thought to be "more" natural than what is conventionally referred to as natural units.

$$k_B = c = \hbar = 1 \tag{A.3}$$

$$\mu_0 = \epsilon_0^{-1} = 4\pi G = 4\pi \tag{A.4}$$

There is also the **rationalized Planck units** where we make the following changes.

$$\mu_0 = \epsilon_0 = 4\pi G = 1 \tag{A.5}$$

The choice between the two basically boils down to if you like the gravitational and electromagnetic forces to have $1/r^2$ (normal) or $1/4\pi r^2$ (rationalized).[6] In these units, all measurements are dimensionless. In reality though, this number can be interpreted as some multiple of the appropriate Planck unit. For example, the basic units are

[6] I like rationalized.

$$l_P = 1.61625 \cdot 10^{-35} \text{ m} \tag{A.6}$$
$$m_P = 2.17643 \cdot 10^{-8} \text{ kg} \tag{A.7}$$
$$t_P = 5.39124 \cdot 10^{-44} \text{ s} \tag{A.8}$$
$$q_P = 1.87554 \cdot 10^{-18} \text{ C} \tag{A.9}$$
$$T_P = 1.41678 \cdot 10^{32} \text{ K} \tag{A.10}$$
$$\tag{A.11}$$

In reality, these all actually equal to 1. However, they are useful in the sense that if I know the length d of something is 2, then I can quickly convert it as

$$d = 2 = 2l_P \approx 3.2 \cdot 10^{-35} \text{ m} \tag{A.12}$$

Of course, you could plug in any of the other basic values but we wanted to convert to meters so we use l_P. There exists an appropriate Planck unit for converting a dimensionless value into any unit or combination of units. For example, if I wanted the associated energy to this distance d in Joules, I would take $d \cdot m_P l_P / t_P^2$ which of course equals d but is in the units of Joules.

It may seem unnatural and even nonphysical use dimensionless numbers for physical values but as we discussed earlier, setting each constant to 1 has very real meaning. Remember that you can still use conventional units because all that Planck units do is give you a means for converting to something more natural if you wish. Do not measure everyday things in seconds!

In a lot of subatomic physics, however, we will use the conventional definition of natural units which is the same as Planck units but with the condition on G relaxed. Sometimes people use the rationalized version and sometimes the classical. In this book, we will use the rationalized version. Because we have relaxed one constraint, the units are no longer dimensionless but we still have only one unique unit. A useful value happens to be electron volts eV with the following relations.

$$1\text{eV} = 1.602 \cdot 10^{-19} \text{ J} \tag{A.13}$$
$$1\text{eV} = 1.785 \cdot 10^{-36} \text{ kg} \tag{A.14}$$
$$1\text{eV}^{-1} = 1.973 \cdot 10^{-7} \text{ m} \tag{A.15}$$
$$1\text{eV}^{-1} = 6.586 \cdot 10^{-16} \text{ J} \tag{A.16}$$
$$1\text{C} = 1.89 \cdot 10^{18} \tag{A.17}$$