QFT Review

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Contents

1 Introduction

What follows is my personal rough lecture notes following the development of the Oxford Quantum Field Theory Lectures given by Prof. John Wheater in the MMathPhys program.

Note: We use the conventions $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $\hbar = c = 1$

2 Relativistic Quantum Mechanics

Quantum mechanics, as we know, provides a great framework to understand particle motion in the quantum non-relativistic regime. In this section, we extend, in an admittedly adhoc sense, this framework to the relativistic regime and find that it does not match the reality we are familiar with. The reason turns out to be that all particles are actually fundamentally fields and we should be quantizing fields, not particles.

2.1 Non-relativistic Quantum Mechanics

Consider the quantization of the Hamiltonian dynamics of a mass m non-relativistic particle. There are two important results of this.

- The 4-vector operator for energy-momentum $\hat{P}^\mu$, in the spacetime basis, takes the form

$$\hat{P}^\mu = -i\partial^\mu \tag{2.1}$$

- The dispersion is

$$\hat{H} = \frac{\hat{p}^2}{2m} \tag{2.2}$$

Combining the two, we have the equation of motion for a non-relativistic quantum particle where $\psi(x^\mu) = \langle x^\mu | \psi \rangle$.

$$i\frac{\partial \psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi \tag{2.3}$$

This has solutions, in the momentum basis,

$$\psi = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} - \epsilon_{\mathbf{p}} t)} \quad \text{where } \epsilon_{\mathbf{p}} = \frac{p^2}{2m} \tag{2.4}$$

As we can see, this equation awkwardly is first order in time and second order in space, an asymmetry that relativity most certainly doesn't allow. We fix this in two ways.

2.2 Klein-Gordon extension

Klein and Gordon fixed this spacetime asymmetry dilemma by simply replacing the dispersion above with the correct relativistic dispersion i.e.

$$\hat{P}_\mu \hat{P}^\mu = -m^2 \tag{2.5}$$

Putting into the results of Hamiltonian quantization, we once again get the equation of motion in the spacetime basis but now with space and time being of the same order.

$$(\Box + m^2)\Psi = 0 \quad \text{where } \Box := \frac{\partial^2}{\partial t^2} - \nabla^2 \tag{2.6}$$

This is the **Klein-Gordon equation** and $\Box$ is called the **D'Alembert operator**. The solution, in momentum basis,

$$\psi = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} - E_{\mathbf{p}}t)} + b_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} + E_{\mathbf{p}}t)} \quad \text{where } E_{\mathbf{p}} = \sqrt{p^2 + m^2} \tag{2.7}$$

Note that there are now two solutions because for each $\mathbf{p}$, both $+E_{\mathbf{p}}$ and $-E_{\mathbf{p}}$ are valid energies.

2.3 Dirac extension

Dirac considered the possibility of a relativistic quantum mechanics that instead of making the equation second order in all spacetime components, an equation that makes it first order in all spacetime components. To keep Lorentz invariance, we can imagine a 4-vector γ^μ such that

$$\gamma^\mu \hat{P}_\mu = m \tag{2.8}$$

which would yield some first order equation for the wavefunction. Because we are still in the relativistic regime, we also still require

$$\hat{P}_\mu \hat{P}^\mu = -m^2 \tag{2.9}$$

Clearly, it doesn't seem to make sense that both of these conditions can simultaneously always hold for arbitrary γ^μ. It turns out however to indeed be possible if we let Ψ now become a vector Ψ_a and γ^μ to be a 4-vector of matrices γ^μ_{ab} where $a, b \in \{1, ...N\}$. Squaring both sides of (2.8), we get that

$$m^2 = \sum_{\mu,\nu} \gamma^\mu \gamma^\nu \hat{P}_\mu \hat{P}_\nu \tag{2.10}$$

$$= \frac{1}{2} \sum_{\mu,\nu} \{\gamma^\mu, \gamma^\nu\} \hat{P}_\mu \hat{P}_\nu \tag{2.11}$$

where $\{\cdot, \cdot\}$ is the anti-commutator. Enforcing the relativistic dispersion $m^2 = -\eta^{\mu\nu} \hat{P}_\mu \hat{P}_\nu$, this gives us that

$$\{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu} \tag{2.12}$$

It turns out that such matrices do indeed exist and so we can write down our final equation of motion which is first order in all spacetime coordinates.

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{2.13}$$

This is called the **Dirac equation** and the γ^μ are called the **gamma matrices**. The solution requires a bit more than just inspection like earlier. Let's do it.

There is only one irreducible form of the matrices satisfying the commutation conditions above and this occurs for $N = 4$. A useful representation are the following matrices.

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1}_2 \\ \mathbb{1}_2 & 0 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \tag{2.14}$$

where σ is the Pauli matrix 3-vector. This also means Ψ has 4 components which we call a **Dirac spinor** which can be split into 2 2-spinors ϕ_1, ϕ_2

$$\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} \tag{2.15}$$

The Dirac equation under these definitions, as before with KG, also gives energies $\pm E_{\mathbf{p}}$ for momentum eigenstate $\mathbf{p}$ but these are now twice degenerate. We can show that, for 2-spinor ϕ,

$$|E_{\mathbf{p}}\rangle \propto \begin{pmatrix} \phi \\ \frac{\hat{\mathbf{p}}\cdot\sigma}{E_{\mathbf{p}}+m}\phi \end{pmatrix} \tag{2.16}$$

$$|-E_{\mathbf{p}}\rangle \propto \begin{pmatrix} -\frac{\hat{\mathbf{p}}\cdot\sigma}{E_{\mathbf{p}}+m}\phi \\ \phi \end{pmatrix} \tag{2.17}$$

We can then define the spin operator in the following natural way.

$$\hat{S}_i = \begin{pmatrix} \frac{1}{2}\sigma_i & 0 \\ 0 & \frac{1}{2}\sigma_i \end{pmatrix} \tag{2.18}$$

This gives us two spin states $\pm$. Taking $\xi^+ = (1,0), \xi^- = (0,1)$, we can show

$$|\pm\rangle \propto \alpha \begin{pmatrix} \xi^\pm \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ \xi^\pm \end{pmatrix} \tag{2.19}$$

This means the eigenspace of a given 3-momentum $\mathbf{p}$ can be decomposed into two more quantum numbers: whether the energy is positive/negative, whether total spin is up/down. The eigenvector for each is given by taking the appropriate energy eigenvector above and letting $\phi = \Xi^\pm$ based on the spin. After normalizing, we can call these positive energy eigenvectors $u^s(\mathbf{p})$ and negative energy eigenvectors $v^s(-\mathbf{p})$ based on the spin s (negative is convention). This gives us the final solution as

$$\psi = \sum_{\mathbf{p}}\sum_{s} a^s_{\mathbf{p}} u^s(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)} + b^s_{\mathbf{p}} v^s(-\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x}+E_{\mathbf{p}}t)} \quad \text{where } E_{\mathbf{p}} = \sqrt{p^2+m^2} \tag{2.20}$$

2.4 Review and Discussion

The quantization of the Hamiltonian dynamics of a particle gives us that $\hat{P}^\mu = -i\partial^\mu$ in the spacetime basis. Based on the dispersion equations provided, we can then directly write down the equation of motion and, from there, general solutions for the wavefunction, the results of which are summarized below.

- **Non-relativistic**

$$\text{Dispersion:} \qquad \hat{H} = \frac{1}{2m}\hat{p}^2 \tag{2.21}$$

$$\text{EOM:} \qquad i\frac{\partial\psi}{\partial t} = -\frac{1}{2m}\nabla^2\psi \tag{2.22}$$

$$\text{Solution:} \qquad \psi = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}-\epsilon_{\mathbf{p}}t)} \tag{2.23}$$

- **Klein Gordon relativistic**

$$\text{Dispersion:} \qquad \hat{P}_\mu \hat{P}^\mu = -m^2 \tag{2.24}$$

$$\text{EOM:} \qquad \Box\Psi = -m^2\Psi \tag{2.25}$$

$$\text{Solution:} \qquad \psi = \sum_{\mathbf{p}} a_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)} + b_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}+E_{\mathbf{p}}t)} \tag{2.26}$$

- **Dirac relativistic**

$$\text{Dispersion:} \qquad \gamma^\mu \hat{P}_\mu = m \quad \text{and } \hat{P}_\mu \hat{P}^\mu = -m^2 \tag{2.27}$$

$$\text{EOM:} \qquad i\gamma^\mu \partial_\mu \psi = m\Psi \quad \text{where } \{\gamma^\mu, \gamma^\nu\} = -2\eta_{\mu\nu} \tag{2.28}$$

$$\text{Solution:} \qquad \psi = \sum_{\mathbf{p}} \sum_{s} a^s_{\mathbf{p}} u^s(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)} + b^s_{\mathbf{p}} v^s(-\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x}+E_{\mathbf{p}}t)} \tag{2.29}$$

Note that in both relativistic solutions, we get infinite negative energy solutions so there is no ground state! How can we ever have a stable system if this is possible? An answer that Dirac proposed was that these negative energy solutions were all filled up with particles by Pauli exclusion principle and that those particles could get excited into positive energy states. The physics of this however requires an understanding of many particle systems which we have not yet encountered.

It turns out that the proper solution to this problem is that we should not be thinking about quantizing the dynamics of particles but instead quantizing the dynamics of fields. In other words, we should not be doing quantum mechanics but **quantum field theory**. *Roll credits*

3 Introductory Field Theory

In this section, we will work out how to deal with both the classical and quantum descriptions of a field. We particularly look at the simple case of a free scalar field which accurately describes free relativistic particles.

3.1 Finite Lattices

A common tool used to understand fields are **finite lattice fields** i.e. fields that take values on some finitely ranged spatially periodic set of points. We only consider cubic lattices in this context. Such structures have two characteristic geometric parameters: lattice constant a and length L. We can then see that we arrive at a field when we take two limits: $a \to 0$ i.e. the **continuum/ultraviolet limit** and $L \to \infty$ i.e. the **infrared limit**. Let's call the joint limit the **field limit**. This is shown pictorially in Figure 1.

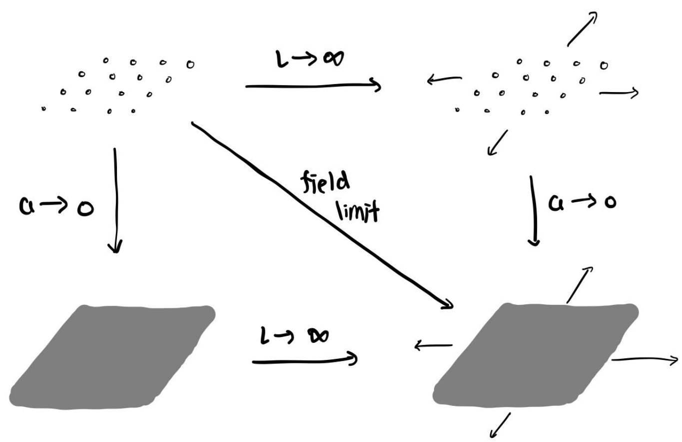

Figure 1: Shown are the two different limits needed to make a finite lattice field an infinite continuum field. Note that taking either one of the limits alone gives us either a finite continuum or infinite lattice. In some sense, we can think of the dynamics of these as equivalent as the Fourier components of a finite continuum take values on a infinite lattice (think discrete FFTs) and Fourier components of an infinite lattice take values on a finite continuum (think Brillouin zones).

The reason we consider $a \neq 0$ is to help our physical intuition so we are not looking at "infinitely close" field values. The reason we consider $L \neq \infty$ is to help our calculus as the density of states and norms are much more well defined (as we know from quantum mechanics anyway).

To go from a classical lattice theory to a quantum field theory, we must both quantize and take the field limit. The relationships between the various ontologies are laid out in Figure 2.

The section will proceed as follows. We will start by understanding the specific field theories we are looking at: free scalar field theories, particularly free $\mathbb{R}$ theories. First, the classical lattice theory is laid out. We then take the field limit to get a classical field theory and then quantize to get a quantum lattice theory. Finally, the quantum field theory is arrived at by taking either of the two and applying the appropriate limit. At the end, given our knowledge, we will directly lay down the free $\mathbb{C}$ quantum field theory.

3.2 Free Scalar Fields

The particular fields we will look at in this chapter have two significant properties.

They are **scalar fields** i.e. they take on scalar values as opposed to maybe spinor, vector, matrix values. In relativistic contexts, it might additionally mean it is frame invariant. For most of the section, we require the scalar to be in $\mathbb{R}$ but, at the end, we consider $\mathbb{C}$. We will find that, physically, scalar corresponds to (relativistic) spin-0 particles and the extension to complex values adds charge.

Also, they are **free fields** or, in other words, they have a linear set of solutions. Two solutions to the equation of motion can be added to get a new one. We will find that this condition physically corresponds to the particles being free.

It turns out though that "free" is not enough to specify a unique dynamical picture but there is a standard one that we will investigate.

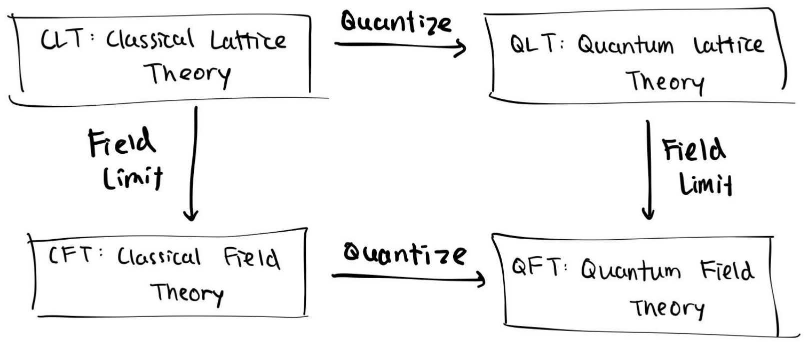

Figure 2: The 4 different ontologies considered for a given field theory.

Ultimately then, we will have a theory which, on the mathematical end, is described by linear scalar field solutions and, on the physical end, describes free relativistic spin-0 particles and find that this matches extraordinarily well with the adhoc Klein-Gordon extension we got earlier but without many of the physical shortcomings.

...MAYBE ADD APPENDIX ON HAMILTONIAN DYNAMICS?

3.3 Classical Dynamics

Start with action, reduce to lagrangian, through legendre transformation get hamiltonian, so we can use canonical quantization ...

particles lattice fields + reciprocal lattices + fourier transforms on lattices fields

3.4 Classical Free $\mathbb{R}$ Lattice Theory

Let the lattice be denoted by $\mathcal{A}$ and NN refer to the nearest neighbor displacements i.e. $\text{NN} = \{\pm a\hat{x}, \pm a\hat{y}, \pm a\hat{z}\}$. From this, we can write down the **Klein-Gordon $\mathbb{R}$ Lagrangian**.

$$L(\mathbf{x}, t) = \frac{1}{2}\left(\frac{\partial \phi}{\partial t}(\mathbf{x}, t)\right)^2 - \frac{1}{2}m^2\phi^2(\mathbf{x}, t) - \frac{\kappa}{4a^2}\sum_{\mu \in \text{NN}} (\phi(\mathbf{x}+\mu, t) - \phi(\mathbf{x}, t))^2 \tag{3.1}$$

Physically, such a Lagrangian is very similar to that of a string or a series of point masses. The first term is the kinetic energy, the second, is the potential energy assuming near equilibrium with frequency m (this notation choice will be clear later), and the third is the coupling to nearby points assuming near equilibrium with coupling constant κ. More generally, there is a energy cost for fluctuation, an energy cost for straying from 0 field value, and an energy cost for differences in adjacent field values.

It's then just a matter of calculus to convert this to a lattice Hamiltonian.

$$H(\mathbf{x},t) = \frac{1}{2}\Pi^2(\mathbf{x},t) + \frac{1}{2}m^2\phi^2(\mathbf{x},t) + \frac{\kappa}{4a^2}\sum_{\mu\in\text{NN}}\left(\phi(\mathbf{x}+\mu,t) - \phi(\mathbf{x},t)\right)^2 \tag{3.2}$$

where $\Pi = \frac{\partial\phi}{\partial t}$ is the canonical momentum lattice field. We now make use of the fact that all these terms are quadratic. By Parseval's theorem, we know that instead of summing the square over all lattice points, we can sum the squares of the Fourier transforms over all reciprocal lattice points. This then gives us the Hamiltonian in momentum space.

$$\begin{aligned}
H(\mathbf{p},t) &= \frac{1}{2}|\Pi(\mathbf{p},t)|^2 + \frac{1}{2}m^2|\phi(\mathbf{p},t)|^2 + \frac{\kappa}{4a^2}\sum_{\mu\in\text{NN}}|\phi(\mathbf{p},t)|^2|e^{i\mathbf{p}\cdot\mu} - 1|^2 && (3.3)\\
&= \frac{1}{2}|\Pi(\mathbf{p},t)|^2 + \frac{1}{2}\underbrace{\left(m^2 + \frac{4\kappa}{a^2}\sum_{i\in\{x,y,z\}}\sin^2\left(\frac{p_i a}{2}\right)\right)}_{=:L_\mathbf{p}^2}|\phi(\mathbf{p},t)|^2 && (3.4)\\
&= \frac{1}{2}|\Pi(\mathbf{p},t)|^2 + \frac{1}{2}L_\mathbf{p}^2|\phi(\mathbf{p},t)|^2 && (3.5)
\end{aligned}$$

Note now that something amazing happened! Instead of working with the field on the real space lattice, we are working on the reciprocal lattice but here, there is no longer near-neighbor coupling. The field at any reciprocal lattice point only has two energy costs: it's own momentum and it's own displacement from equilibrium. The only cost of this was that the frequency of vibration at each reciprocal lattice point is not constant anymore but this is a perfectly fine trade off if we are looking at just one point. We now know that the equation of motion of each field value will be independent of all the others i.e. these are the normal modes of the system. Not only this, but each of these modes has the Hamiltonian of a simple harmonic oscillator! The Klein-Gordon Lagrangian then describes a set of independent SHOs.

should I include EOM and solutions?

3.5 Classical Free $\mathbb{R}$ Field Theory

Now let's take the field limit. Firstly, this changes our lattice sums into integrals.

$$\sum_{\mathbf{x}\in\mathbb{A}} a^3 \to \int \mathrm{d}\mathbf{x} \tag{3.6}$$

$$\sum_{\mathbf{p}\in\mathbb{G}} \frac{1}{L^3} \to \frac{1}{(2\pi)^3}\int \mathrm{d}\mathbf{p} \tag{3.7}$$

These sums only come out in the expression of the full Lagrangian and Hamiltonian though. We can also observe the effects, on the individual Lagrangian and Hamiltonian densities. Carrying out the limits, we notice that we get the following under the limits.

$$L(\mathbf{x},t) = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}(\mathbf{x},t)\right)^2 - \frac{1}{2}m^2\phi^2(\mathbf{x},t) - \frac{1}{2}\kappa|\nabla\phi(\mathbf{x},t)|^2 \tag{3.8}$$

$$H(\mathbf{x},t) = \frac{1}{2}\Pi^2(\mathbf{x},t) + \frac{1}{2}m^2\phi^2(\mathbf{x},t) + \frac{\kappa}{2}|\nabla\phi(\mathbf{x},t)|^2 \tag{3.9}$$

$$H(\mathbf{p},t) = \frac{1}{2}|\Pi(\mathbf{p},t)|^2 + \frac{1}{2}(m^2 + \kappa p^2)|\phi(\mathbf{p},t)|^2 \tag{3.10}$$

The first thing to notice is that couplings in continuum fields are given by derivative terms in the Lagrangian/Hamiltonian densities. Secondly, if we want our Lagrangian to be a *relativistic* scalar i.e. Lorentz invariant, then we can do this by setting $\kappa = 1$ which gives us

$$L = -\frac{1}{2}|\partial\phi|^2 - \frac{1}{2}m^2\phi^2 \tag{3.11}$$

$$H(\mathbf{x},t) = \frac{1}{2}\Pi^2(\mathbf{x},t) + \frac{1}{2}m^2\phi^2(\mathbf{x},t) + \frac{1}{2}|\nabla\phi(\mathbf{x},t)|^2 \tag{3.12}$$

$$H(\mathbf{p},t) = \frac{1}{2}|\Pi(\mathbf{p},t)|^2 + \frac{1}{2}E_{\mathbf{p}}|\phi(\mathbf{p},t)|^2 \quad \text{where } E_{\mathbf{p}}^2 = m^2 + p^2 \tag{3.13}$$

The Lagrangian density is particularly pretty here. We then get the equation of motion for our field.

$$\frac{\partial^2\phi}{\partial t^2} = -E_{\mathbf{p}}^2\phi \tag{3.14}$$

which can then be converted back into real space to get

$$\Box\phi = -m^2\phi \tag{3.15}$$

This is exactly the Klein-Gordon equation if we let m represent the mass. We now have a classical field theory that gives the same equation of motion of the Klein-Gordon equation. We still haven't fixed any of the problems before though but the theory is no longer adhoc; it has a Lagrangian and a Hamiltonian. This allows us to naturally quantize it which we will find out is the resolution to our problems. Before jumping into the quantum field theory though, let's back up and first do the quantum lattice theory.

3.6 Quantum Free $\mathbb{R}$ Lattice Theory

Canonical quantization usually has 3 parts.

- Choosing a dynamical picture (Schrodinger, Heisenberg, etc.)
- Turning relevant observables into operators and defining relations. Usually, this amounts to equating the Poisson bracket to the commutator (we will soon see divergences from this) and writing the operator version of the Hamiltonian function
- Defining the state space. This is heavily regulated by the operator relations as they are commonly labeled by the eigenspaces of chosen operators.

For the dynamical picture, we can immediately choose Heisenberg. The reason for this is as follows. The basis of the quantum Hilbert state space approximately resembles the classical state space. Given that our classical state space consists of every possible field configuration, it quickly becomes intractable to track the amplitudes associated with each configuration. Therefore, we avoid messing with an innumerable mess of amplitudes and put the time dependence in our nice compactly notated operators.

The second part translates to the following two equations where we work in the normal mode basis.

$$[\hat{\phi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] = i\delta_{\mathbf{p}\mathbf{p}'} \tag{3.16}$$

$$[\hat{\phi}_{\mathbf{p}}, \hat{\phi}_{\mathbf{p}'}] = 0 = [\hat{\Pi}_{\mathbf{p}}, \hat{\Pi}_{\mathbf{p}'}] \tag{3.17}$$

$$\hat{H} = \frac{1}{L^3} \sum_{\mathbf{p} \in \mathbb{G}} \frac{1}{2} |\hat{\Pi}_{\mathbf{p}}|^2 + \frac{1}{2} L_{\mathbf{p}}^2 |\hat{\phi}_{\mathbf{p}}|^2 \tag{3.18}$$

As discussed earlier, the normal modes are described by many independent SHOs, one for each $\mathbf{p}$ and from our knowledge of quantum mechanics, we know that quantum SHOs are expressed more naturally with ladder operators.

$$a_{\mathbf{p}} := \sqrt{\frac{L_{\mathbf{p}}}{2}} \left(\hat{\phi}_{\mathbf{p}} + \frac{i}{L_{\mathbf{p}}} \hat{\Pi}_{\mathbf{p}} \right) \tag{3.19}$$

$$\implies \hat{\phi}_{\mathbf{p}} = \frac{1}{\sqrt{2L_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger) \tag{3.20}$$

$$\hat{\Pi}_{\mathbf{p}} = i\sqrt{\frac{L_{\mathbf{p}}}{2}} (a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}) \tag{3.21}$$

We can then write all of our relations above with just ladder operators.

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}^\dagger] = \delta_{ij} \tag{3.22}$$

$$[a_{\mathbf{p}}, a_{\mathbf{p}'}] = 0 = [a_{\mathbf{p}}^\dagger, a_{\mathbf{p}'}^\dagger] \tag{3.23}$$

$$\hat{H} = \frac{1}{L^3} \sum_{\mathbf{p} \in \mathbb{G}} L_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2} \right) \tag{3.24}$$

Second part of quantization is now complete. All that remains is the state space. The state space can be completely defined by one statement: there exists a state $|\Omega\rangle$ called the **vacuum state** such that

$$a_{\mathbf{p}} |\Omega\rangle = 0 \quad \forall \mathbf{p} \in \mathbb{G} \tag{3.25}$$

By straightforward analysis very similar to the usual quantum mechanical treatment of SHOs, we can then conclude that every state in the state space is defined by which ladder operators were applied and how many times. In other words, a vector in the state space is given by

$$|n(\mathbf{p})\rangle = \prod_{\mathbf{p} \in \mathbb{G}} (a_{\mathbf{p}}^\dagger)^n (\mathbf{p}) |\Omega\rangle \quad \text{where } n : \mathbb{G} \to \mathbb{N}_0 \tag{3.26}$$

We can enumerate all states by specifying all functions $n : \mathbb{G} \to \mathbb{N}_0$. Such a space is called **Fock space**. We have thus finished quantizing.

There are, of course, a few things left.

- Time evolution of the system
- Physical interpretation of system. Right now, we have just arrived at some abstract quantum picture from canonically quantizing the Klein-Gordon field.

Both of the above become much clearer once we find the spectral decomposition of $\hat{H}$. It's not hard to play with the ladder algebra like in quantum mechanics to arrive at the relation

$$\hat{H}|n\rangle = L_\Omega + \frac{1}{L^3}\sum_{\mathbf{p}\in\mathbb{G}} n_{\mathbf{p}} L_{\mathbf{p}} \quad \text{where } \hat{H}|\Omega\rangle L_\Omega|\Omega\rangle \tag{3.27}$$

There are two physical comments to note here. Firstly, we can show

$$L_\Omega = \frac{1}{L^3}\sum_{\mathbf{p}\in\mathbb{G}} \frac{1}{2} L_{\mathbf{p}} \tag{3.28}$$

i.e. the vacuum energy L_Ω is simply the zero-point energy of all the modes. This offset doesn't really have many physical ramifications on our states as they are all offset by the same value which we know is inconsequential dynamically.[1]

Secondly, it seems to make a lot of physical sense now that the state $|n(\mathbf{p})\rangle$ corresponds to a many particle system with **occupations** $n(\mathbf{p})$. Energetically, assuming the $\mathbf{p}$ state has energy $L_{\mathbf{p}}$, (3.27) is exactly what we'd expect for such a system. Indeed, this is the case and an appropriate physical interpretation of our new quantum system. Given such an interpretation and remembering $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ gives the number of particles of momentum $\mathbf{p}$, it makes sense to define the following operators.

$$\hat{\mathbf{P}} = \frac{1}{L^3}\sum_{\mathbf{p}\in\mathbb{G}} \mathbf{p} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{3.29}$$

$$\hat{N} = \frac{1}{L^3}\sum_{\mathbf{p}\in\mathbb{G}} a^\dagger_{\mathbf{p}} a_{\mathbf{p}} \tag{3.30}$$

which correspond to the total momentum operator and total number operator respectively. Some straightforward calculation gives the relations

$$[\hat{N}, \hat{H}] = 0 \tag{3.31}$$

$$[\hat{\mathbf{P}}, \hat{H}] = 0 \tag{3.32}$$

so both total momentum and total particle number is conserved.

The final step then is to calculate time evolution which, for a general observable $\hat{A}$, is given, in the Heisenberg picture, by

$$i\frac{\partial \hat{A}}{\partial t} = [\hat{A}, \hat{H}] \tag{3.33}$$

[1] Other than in general relativity where the absolute energy density and curvature are directly related. This vacuum energy density then comes out in the form of a cosmological constant. In practice though, all calculations of zero-point energy are many orders of magnitude off the observed constant so this is not the full story of the vacuum.

Everything can be connected to the ladder operators so let's find its evolution. Due to the linear nature of both the above equation and the ladder operator definition, we can let $\hat{A} = a_{\mathbf{p}}$ in the above equation. Carrying out the calculus, we get

$$a_{\mathbf{p}}(t) = e^{-iL_{\mathbf{p}}t} a_{\mathbf{p}} \tag{3.34}$$

Ultimately, our goal is to find out how this field $\hat{\phi}(\mathbf{x}, t)$ evolves. Plugging the ladder operator back in and conducting the Fourier transform, we arrive at our general solution.

$$\hat{\phi}(\mathbf{x}, t) = \frac{1}{L^3} \sum_{\mathbf{p} \in \mathbb{G}} \frac{1}{\sqrt{2L_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i(-L_{\mathbf{p}}t + \mathbf{p}\cdot\mathbf{x})} + a_{\mathbf{p}}^\dagger e^{i(L_{\mathbf{p}}t + \mathbf{p}\cdot\mathbf{x})} \right) \tag{3.35}$$

insert physical interpretation

3.7 Quantum Free $\mathbb{R}$ Field Theory

Just like in the classical situation, there is not actually much difference between the lattice and field situation because a and L barely show up! We can just reuse the equations from the previous section with 3 important changes.

$$\sum_{\mathbf{x} \in \mathbb{A}} a^3 \to \int \mathrm{d}\mathbf{x} \tag{3.36}$$

$$\sum_{\mathbf{p} \in \mathbb{G}} \frac{1}{L^3} \to \frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{p} \tag{3.37}$$

$$L_{\mathbf{p}} \to m^2 + \kappa p^2 \tag{3.38}$$

$$= E_{\mathbf{p}} \quad \text{if relativistic} \tag{3.39}$$

All the physical interpretations and results stay the same with the minor difference now that the occupations n are functions from all momentum space $\mathbb{R}^3$ instead of just the reciprocal lattice $\mathbb{G}$ as expected. There is one particular issue that arises though. Consider our new relativistic vacuum energy.

$$L_\Omega \to E_\Omega = \frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{p} \frac{1}{2} E_{\mathbf{p}} \tag{3.40}$$

We note that

$$E_\Omega = \frac{1}{(2\pi)^3} \int \mathrm{d}\mathbf{p} \frac{1}{2} \sqrt{m^2 + p^2} \to \infty \tag{3.41}$$

Our vacuum energy now diverges! It diverges because we are considering arbitrarily high $\mathbf{p}$, a symptom of the ultraviolet part of the field limit. This is called an **ultraviolet divergence** and indicates that at higher energies, this quantum field theory must fall apart. High $\mathbf{p}$ corresponds to small wavelengths so this indication can alternatively be phrased as at very small scales, quantum field theory must fall apart.

both phi and pi satisfy KG equation

3.8 Quantum Free $\mathbb{C}$ Field Theory

3.9 Review and Discussion

4 Some QFT Calculus

4.1 Normal Ordering

4.2 Propagators

APPENDIX ON LAGRANGIAN AND HAMILTONIAN FORMALISMS