But what really is symmetry? (Apparent vs. Actual Structure)

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As I see it, symmetry is an artifact of us considering apparent instead of actual structures when analyzing systems. So let's get into it.

Say we try to describe the dynamics of a particle in space. We draw out a coordinate system/vector space with some origin and specify that this system has translational symmetry by saying that we can shift this origin. Alternatively, someone could argue that we shouldn't put in an origin at all for space in reality has no origin. In fact the appropriate structure for describing this situation is called an **affine space**, basically a vector space without an origin. In this space, there is no translational symmetry. It is inherent in the structure! Affine spaces only have notions of displacements, NOT position. So one may ask the question:

- Is physical space an affine space or a vector space with translational symmetry?

The answer is both! Modeling the world comes with many choices and based on the structure, there are different explicit symmetries. Why then would we ever work on a redundant or, in harsher terms, wrong structure? Why use a vector space structure and manually insert symmetries instead of just working on an affine space? Well the answer is: it's just easier. Even when we do work with affine spaces, we often construct a vector space to work on it anyway so we can simply start there. Translational symmetry only FURTHER simplifies the problem for us so affine spaces are actually double the work in some sense. But this brings us to the core of what symmetry is:

$$\boxed{\text{Actual Structure} = \frac{\text{Apparent Structure}}{\text{Symmetry}}} \tag{1}$$

In our case,

$$\text{Affine/Physical Space} = \frac{\text{Vector Space}}{\text{Translation}} \tag{2}$$

Although this may seem a hand-wavey statement, we know symmetry implies a certain equivalence under some transformation i.e. specifies an equivalence relation. This means we can take our apparent structure (the full set of states), find equivalence classes under this relation i.e. construct the quotient set and arrive at our actual structure. Hence the statement above is formally rigorous.

To really understand why this is so insightful though, let's look at some more examples.

- **Quantum Field Theory**: In quantum mechanics, people often get their first experience to the **exchange symmetry** with the Hydrogen atom. Here, we specify that two electrons are **indistinguishable** meaning they represent the same physical state if swapped. This fact however, is manually and awkwardly added in an ad-hoc manner. Why? Because the actual structure of the electrons is a field! Not particles.

$$\text{Quantum Field} = \frac{\text{Many Quantum Particles}}{\text{Exchange}} \tag{3}$$

- **Topology**: How do we usually phrase the definition of topology? As some geometric object that is defined only up to deformations. This is because the concept of deformation and geometry are easy to grasp but we can't even begin to think about pure topological structures!

$$\text{Topology} = \frac{\text{Geometry}}{\text{Deformation}} \tag{4}$$

- **Conformal Manifolds**: Conformal spaces are spaces that only have a notion of angle. They are often however phrased as

$$\text{Conformal Space} = \frac{\text{Space}}{\text{Angle-Preserving Maps}} \tag{5}$$

  For manifolds of greater than 3 dimension, we often define a conformal manifold as

$$\text{Conformal Manifold} = \frac{\text{Riemannian Manifold}}{\text{Scaling}} \tag{6}$$

  i.e. we give a notion of angle by giving a notion of dot product and taking away the magnitude.

- **Relativity**: We can succinctly phrase relativity as the following.

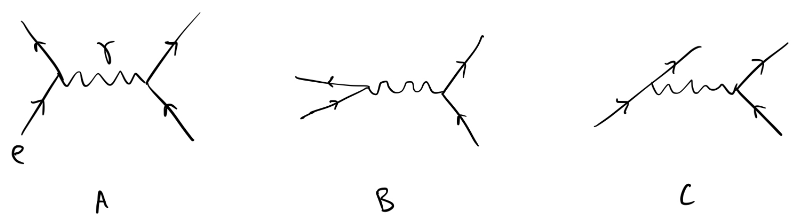
$$\text{Physics} = \frac{\text{Our Models}}{\text{Moving the Observer}} \tag{7}$$

- **Notation**: Many times, our notation for certain structures has MORE structure than what it wishes to represent. For example, a set $\{a, b, c\}$ has no order but it must have one when written down. We must then specify that this phrase has a symmetry under permutation of its elements.

$$\text{Structure} = \frac{\text{Notation}}{\text{Notational Redundancy}} \tag{8}$$

Although seemingly trivial, it can become nontrivial in cases like Feynmann diagrams. For example, see the diagram below. It is clear that A and B represent the same process of an electron and positron annihilating into a photon but the third process C represents an electron turning into an electron and photon. We must make it clear then that a diagram is only partially deformable when drawn out. It turns out however that the physical theory has a **crossing symmetry** which makes the amplitudes for C equivalent to A and B. This means we can absorb this symmetry by simply releasing our restrictions on the notation. In this case, we got lucky where the interpretation without symmetry was simpler than with symmetry which is why we rarely discuss the crossing symmetry.

$$\text{Unrestricted Diagrams} = \frac{\text{Restricted Diagrams}}{\text{Crossing Symmetry}} \tag{9}$$



Hopefully, this fairly diverse set of examples drives home this understanding of symmetry as marking the gap between apparent and actual structures.